

FJRW-RINGS AND LANDAU-GINZBURG MIRROR SYMMETRY

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ABSTRACT. In this article, we study the Berglund–Hübsch transpose construction W^T for invertible quasi-homogeneous potential W . We introduce the dual group G^T and establish the state space isomorphism between the Fan–Jarvis–Ruan–Witten A -model of W/G and the orbifold Milnor ring B -model of W^T/G^T . Furthermore, we prove a mirror symmetry theorem at the level of Frobenius algebra structure for G^{\max} . Then, we interpret Arnol'd strange duality of exceptional singularities W as mirror symmetry between $W/\langle J \rangle$ and its strange dual W^{SD} .

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1. INTRODUCTION

During the last twenty years, mirror symmetry has been a driving force for some of the developments in geometry and physics. In this article, we add to this development a version of mirror symmetry purely in the Landau-Ginzburg / singularity setting, i.e. we produce a mirror LG theory to a given LG theory. This version of mirror symmetry is inspired by an early proposal of Berglund–Hübsch [BH] for invertible singularities and the recent development of quantum singularity theory / LG topological string of Fan–Jarvis–Ruan–Witten [FJR1]. Compared to the other forms of mirror symmetry such as Calabi-Yau via Calabi-Yau and toric via LG, our version has the benefit of not having any poorly behaved exceptional cases. These exceptional cases hindered the formulation of mathematical conjectures in the other forms of mirror symmetry. This makes the present (LG via LG) model of mirror symmetry attractive for future study. Historically, this version has been used in physics to verify geometric mirror symmetry (CY via CY) through the conjectured LG / CY correspondence. LG via LG is certainly more general, since the LG-orbifold theories under consideration do not have to correspond to Calabi-Yau manifolds. Even if they do correspond to Calabi-Yau manifolds (orbifolds), they are not necessarily the Gorenstein orbifolds where a mathematical proof was established by Batyrev [B]. The author has been informed that this generality, combined with a proof of LG / CY correspondence, has been exploited by Chiodo–Ruan [CR] to generalize Batyrev’s theorem in Calabi-Yau hypersurface of Gorenstein weight projective spaces.

LG via LG mirror symmetry is not a new idea. As indicated, it was an important physical tool to verify Calabi-Yau mirror symmetry in the early investigations of this phenomenon. Throughout the literature, a striking construction was Berglund–Hübsch’s trasposed potential. Let me briefly review this construction, which restricts consideration to so-called *invertible* singularities. A quasi-homogeneous polynomial $W = \sum_{i=1}^s c_i \prod_{j=1}^N z_j^{a_{ij}}$ is invertible if $s = N$ (i.e. if the number of monomials equals the number of variables). Then the matrix $A_W = (a_{ij})$ of exponents is square, and the matrix A_W^T defines another quasi-homogeneous invertible potential – the *Berglund–Hübsch dual* – which we denote by $W^T = \sum_{i=1}^s c_i \prod_{j=1}^N z_j^{a_{ji}}$. Berglund–Hübsch proposed almost twenty years ago [BH] that (W, W^T) forms a mirror pair. It was known that one must consider orbifold LG-models $(W/G, W^T/G^T)$ for this proposition to have any chance to be correct. In the literature, the construction of the dual group G^T is known in many cases (e.g. for the Fermat Quintic), and we present a general construction in Section 3.1. We should emphasize that the subject of LG via LG mirror symmetry was never fully developed in physics because (i) a construction of the A -model was absent, and (ii) although the orbifold B -model state space was given by Intriligator–Vafa [IV], the ring structure was still lacking. The first problem was solved recently by Fan–Jarvis–Ruan–Witten [FJR1]–[FJR3] with the establishment of quantum singularity theory / LG-topological string. As for the second problem, Kaufmann wrote down the multiplication in many cases and proposed a general recipe [Ka1]–[Ka3]. Guided by his recipe, we wrote down a multiplication for non-degenerate invertible singularities W and $G \subset SL$. Our definition of multiplication has an important restriction not present in Kaufmann’s recipe, namely that the B -model orbifold group should be a subgroup of $SL_N\mathbb{C}$. This is dual to the fact that in Fan–Jarvis–Ruan–Witten’s construction, every admissible A -model orbifold group must contain the exponential grading operator J .

With both the A -model and the B -model established, we can study the LG via LG mirror symmetry. Our main theorems are:

Theorem 1.1. *Let W be a non-degenerate invertible potential and G a group of diagonal symmetries of W . There is a bi-graded isomorphism of vector spaces*

$$\mathcal{H}_{W,G} \cong \mathcal{Q}_{W^T, G^T},$$

where $\mathcal{H}_{W,G}$ is the FJRW A -model of (W, G) and \mathcal{Q}_{W^T, G^T} is the orbifold B -model of (W^T, G^T) .

Theorem 1.2. *Let W be a non-degenerate invertible potential and G^{\max} its maximal group of diagonal symmetries. There is an isomorphism of Frobenius algebras*

$$\mathcal{H}_{W, G^{\max}} \cong \mathcal{Q}_{W^T},$$

where \mathcal{Q}_{W^T} is the unorbifolded B -model of W^T .

Invertible singularities include, for example, Arnol'd's list of simple, unimodal and bimodal singularities [AGV]. Theorem 1.2 has already been proven for the simple and parabolic singularities [FJR1] and the unimodal and bimodal singularities [KP+]. The 14-families of exceptional (unimodal) singularities exhibit the famous *Arnol'd strange duality*. There have been attempts to explain strange duality by relating exceptional singularities to K3-surfaces. It is possibly more natural to consider it from the LG via LG mirror symmetry perspective. For example, we apply Theorem 1.2 to show that strange duality indeed agrees with LG via LG mirror symmetry.

Corollary 1.3. *Let W be one of Arnol'd's 14 exceptional singularities with strange dual W^{SD} , and J its exponential grading operator. Then*

$$\mathcal{H}_{W, \langle J \rangle} \cong \mathcal{Q}_{W^{SD}}.$$

i.e. the LG A -model for W orbifolded by J is isomorphic (as a Frobenius algebra) to the unorbifolded LG B -model of W^{SD} .

We should mention that Theorem 1.1 specializes in the case of $G = G^{\max}$ to the main result of Kreuzer in [K]. That work considers only a single grading, and appeals to physically motivated ‘twist selection rules’ to argue that the mirror map is degree-preserving. We clarify the physical picture, and establish our theorems in the most general context (bi-grading, dual group, Frobenius algebra structure) in order to set the stage for the future applications of LG-mirror symmetry. As we mentioned previously, one important application is the CY via CY mirror symmetry of CY-hypersurfaces of non-Goreinstein weighted projective space which is exploited by Chiodo-Ruan.

Another important application is the integrable hierarchies problem. Recall that there is a semi-simple Frobenius manifold theory for the unorbifolded B -model of W^T due to Saito and the high-genus theory by Givental. Theorem 1.2 naturally suggests the following conjecture

Conjecture. *Let W be a non-degenerate invertible potential and G^{\max} be its maximal group of diagonal symmetries. Then the full FJRW-theory of W/G is isomorphic to Saito–Givental theory of W^T .*

In many cases, the Saito–Givental theory of W^T is expected to satisfy certain integrable hierarchies. The study of these examples leads to a generalization of Witten’s famous ADE integrable hierarchies conjecture solved by Fan–Jarvis–Ruan [FJR1]. We refer the interested readers to [R2] for the details.

The paper is organized as follows.

1.1. Organization of paper. We present some basic notions regarding invertible potentials in Section 1.3, including Kreuzer–Skarke’s classification of invertible potentials.

In Section 2 we review the construction of the FJRW A -model Frobenius algebra, as well as the orbifold B -model state space of Intriligator–Vafa. We introduce a multiplication on the orbifold B -model and show that this multiplication respects a suitably shifted version of the bi-grading of Intriligator–Vafa.

In Section 3 we prove an LG-via-LG mirror symmetry result, after introducing a suitable notion of duality between the symmetry groups of Berglund–Hübsch dual potentials.

In Section 4, we show that there is an isomorphism of Frobenius algebras between the maximally orbifolded A -model of a singularity and the unorbifolded B -model of the Berglund–Hübsch dual. Finally, we demonstrate a relation between Arnol'd’s strange duality and the LG-via-LG mirror symmetry discussed in this paper.

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1.3. Preliminaries on Invertible Potentials. Consider an invertible non-degenerate quasihomogeneous potential

$$W = \sum_{i=1}^N c_i \prod_{j=1}^N X_j^{a_{ij}}.$$

The exponent matrix $A = (a_{ij})$ encodes the singularity, modulo the coefficients c_i of the monomials.

The charges (or weights) q_i are determined by the condition that

$$A \begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \quad (1)$$

Remark. Since A is invertible, the c_i may be absorbed by rescaling the variables. In what follows, we will take $c_i = 1$ without loss of generality.

If the matrix A_W is square, its transpose A_W^T will also correspond to a quasi-homogeneous polynomial, which we denote by W^T .

Non-degeneracy of W requires the charges to be uniquely determined, so $\det A \neq 0$.

We write

$$A^{-1} = \left(\begin{array}{c|c|c|c} \rho_k & \rho_2 & \cdots & \rho_N \end{array} \right), \quad \text{with column vectors} \quad \rho_k = \begin{bmatrix} \varphi_1^{(k)} \\ \vdots \\ \varphi_N^{(k)} \end{bmatrix} \quad (2)$$

Then each ρ_k defines a symmetry of W via

$$\rho_k X_j = \exp(2\pi i \varphi_j^{(k)}) X_j.$$

We will abuse notation and use the same symbol to denote the symmetry and the column vector.

Remark. Suppose $g \in (\mathbb{C}^*)^N$ is a diagonal symmetry of W , with $gX_k = \exp(2\pi i g_k) X_k$. g preserving W is equivalent to

$$A \begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix} \in \mathbb{Z}^N,$$

So the phase vector $(g_1, \dots, g_N)^T$ is a linear combination of the columns of A^{-1} . This implies that the ρ_k generate the group G_{\max} of diagonal symmetries of W . For any $g \in G^{\max}$, we can write $g = \prod_{i=1}^N \rho_i^{\alpha_i}$.

Remark. The group G^{\max} is non-trivial, as it contains the *exponential grading operator* J , which acts on X_k with phase q_k .

Multiplying Equation (1) by A^{-1} , we see that J is given by

$$J = \prod_{i=1}^N \rho_i \quad i.e. \quad |J| = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad (3)$$

In [KS], Kreuzer and Skarke prove that an invertible potential is non-degenerate if and only if it can be written as a sum of (decoupled) invertible potentials of one of the following three types, which we will refer to as *atomic types*:

$$W_{\text{Fermat}} = X^a.$$

$$W_{\text{loop}} = X_1^{a_1} X_2 + X_2^{a_2} X_3 + \cdots + X_{N-1}^{a_{N-1}} X_N + X_N^{a_N} X_1.$$

$$W_{\text{chain}} = X_1^{a_1} X_2 + X_2^{a_2} X_3 + \cdots + X_{N-1}^{a_{N-1}} X_N + X_N^{a_N}.$$

Although this classification allows for terms $X_k X_k + 1$ (i.e. $a_k = 1$), we will only consider the case $a_i \geq 2$ so that the charges satisfy $q_i \leq \frac{1}{2}$, as this condition is necessary for the construction of the FJRW A -model.

Remark. The proof of Theorem 4.1 Chain potentials is valid only if $a_N > 2$, so that all charges are strictly less than $\frac{1}{2}$.

It is clear that the transpose construction W^T preserves the above types. Our arguments will rely heavily on an understanding of these ‘atomic’ potentials and their symmetry groups, and we recall without proof some elementary facts from [K] below. Because the Fermat potential is particularly straightforward, our discussion focuses on Loops and Chains.

Notation. We use the Dirac delta:

$$\delta_{\alpha, \beta} := \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{else.} \end{cases}$$

Also

$$\delta_i^{\text{even}} := \begin{cases} 1 & \text{if } i \text{ is even,} \\ 0 & \text{else,} \end{cases}$$

with δ_i^{odd} defined similarly.

We now recall without proof some facts from [K] which will be useful in the sequel.

The following lemma facilitates computation of the phase of a given symmetry on a variable X_j .

Lemma 1.4. *Let $W \in \mathbb{C}[X_1, \dots, X_N]$ be a non-degenerate invertible potential of atomic type, with exponent matrix A_W and generators of G^{\max} given by ρ_1, \dots, ρ_N corresponding to the columns of A_W^{-1} . Let $g = \left(\prod_{i=1}^N \rho_i^{\alpha_i} \right)$, with $0 \leq \alpha_i < a_i$. For $j \in \{1, \dots, N\}$ with X_j not fixed by gJ ,*

$$\Theta_j^{gJ} = \sum_{i=1}^N (\alpha_i + 1) \varphi_j^{(i)}.$$

i.e. *The phase of gJ on X_j is given by the algebraic sum of the phases of the ρ_i on X_j , without the need to reduce this sum modulo 1.*

If X_j is fixed by gJ , $\Theta_j^{gJ} = 0$ although the algebraic sum of phases may equal either 0 or 1 (and these cases can be explicitly identified).

The following lemma gives explicit generators over \mathbb{C} for the Milnor ring of a loop or chain potential.

Lemma 1.5.

- *The Milnor ring $\mathcal{Q}_{W_{\text{loop}}}$ for a loop potential is generated over \mathbb{C} by $\{\prod_{i=1}^N X_i^{\alpha_i} \mid 0 \leq \alpha_i < a_i\}$, and has dimension $\prod_{i=1}^N a_i$.*

- The Milnor ring $\mathcal{Q}_{W_{chain}}$ for a chain potential is generated over \mathbb{C} by $\{\prod_{i=1}^N X_i^{\alpha_i} \mid 0 \leq \alpha_i < a_i\}$ subject to the condition that the largest set $\{1, \dots, s\}$ of consecutive indices for which $\alpha_i = \delta_i^{odd}(a_i - 1)$ has an even number of elements (possibly zero). Its dimension is $\sum_{i=odd} (a_i - 1) \prod_{j=i+1}^N a_j$, where we interpret the empty product as equal to 1.

For $g \in G^{max}$, the next lemma identifies the G^{max} -invariants in $\mathcal{Q}_{Fix(gJ)}$. The lemma after next one identifies the symmetry groups of atomic invertible potentials.

Lemma 1.6.

- Let W_{loop} be a loop potential. Then G_W^{max} has order $\prod_{i=1}^N a_i - (-1)^N$.
If N is even, any symmetry g of W_{loop} may be written

$$g = \prod_{i=1}^N \rho_i^{\alpha_i} \quad \text{with} \quad 0 \leq \alpha_i < a_i.$$

This presentation is unique, except in the case of J^{-1}

$$J^{-1} = \prod_{i \text{ even}} \rho_i^{a_i-1} \quad \text{and} \quad J^{-1} = \prod_{i \text{ odd}} \rho_i^{a_i-1}$$

If N is odd, any symmetry $g \neq J^{-1}$ of W_{loop} may be written uniquely as

$$g = \prod_{i=1}^N \rho_i^{\alpha_i} \quad \text{with} \quad 0 \leq \alpha_i < a_i.$$

- Let W_{chain} be a chain potential. Then G_W^{max} has order $\prod_{i=1}^N a_i$, and any $g \in G_W^{max}$ may be written uniquely as

$$g = \prod_{i=1}^N \rho_i^{\alpha_i} \quad \text{with} \quad 0 \leq \alpha_i < a_i.$$

Remark. Lemmas 1.5, and 1.6 combine to show that for loop and chain potentials, image the map

$$\begin{aligned} \mathcal{Q}_{W_{chain}}^T &\longrightarrow G_W \\ \prod_{i=1}^N Y_i^{\alpha_i} dY_i &\longmapsto \left(\prod_{i=1}^N \rho_i^{\alpha_i} \right) J \end{aligned}$$

is the collection of group elements with *even dimensional* fixed loci. The map is injective for chains, and simply ramified over J^{-1} for loops when N is even.

Lemma 1.7.

- For a loop potential W_{loop} , the only symmetry gJ with non-trivial fixed locus is $gJ = \text{id}$, which has fixed locus \mathbb{C}^N . Generators of the G^{max} invariants as a \mathbb{C} -vector space are given by

$$\mathcal{Q}_{Fix(gJ)}^{G^{max}} = \begin{cases} \emptyset & \text{if } gJ = \text{id}, \text{ and } N \text{ is odd.} \\ \left\{ \prod_{i=1}^N X_i^{\delta_i^{even}(a_i-1)} dX_i, \prod_{i=1}^N X_i^{\delta_i^{odd}(a_i-1)} dX_i \right\} & \text{if } gJ = \text{id}, \text{ and } N \text{ is even.} \\ 1 & \text{else.} \end{cases}$$

- For a chain potential, W_{chain} , if a symmetry gJ fixes X_t , it must fix $\{X_t, \dots, X_N\}$. $\text{Fix } gJ = \{X_t, \dots, X_N\}$ implies $g = \prod_{i=1}^N \rho_i^{\alpha_i}$ has $\alpha_i = \delta_{N-i}^{even}(a_i - 1)$ for $i \geq t$, and this relation does not hold for $i = t - 1$

The G^{max} -invariants in $\mathcal{Q}_{Fix(gJ)}$ are generated by

$$\mathcal{Q}_{Fix(gJ)}^{G^{max}} = \begin{cases} \emptyset & \text{if } \text{Fix}(gJ) = \{X_t, \dots, X_N\} \text{ is odd-dimensional.} \\ \prod_{i=t}^N X_i^{\delta_{i-t}^{even}(a_i-1)} dX_i, & \text{if } \text{Fix}(gJ) = \{X_t, \dots, X_N\} \text{ is even-dimensional,} \\ 1 & \text{if } \text{Fix}(gJ) = \emptyset \end{cases}$$

2. THE A AND B MODELS

2.1. FJRW A -model. Let W be a non-degenerate quasi-homogeneous polynomial in the variables x_1, x_2, \dots, x_N with weights q_1, q_2, \dots, q_N respectively. Non-degeneracy requires that these weights are uniquely determined by the condition that each monomial in W has total weight 1, and that W has an isolated singularity at the origin.

The central charge of W is defined to be

$$\hat{c} := \sum_{j=1}^N (1 - 2q_j).$$

The Jacobian ideal \mathcal{J} is defined by

$$\mathcal{J} = \left\langle \frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \dots, \frac{\partial W}{\partial x_N} \right\rangle.$$

The Milnor ring \mathcal{Q}_W is given by

$$\mathcal{Q}_W := \mathbb{C}[x_1, x_2, \dots, x_N] / \mathcal{J}$$

together with the residue pairing. \mathcal{Q}_W is a finite dimensional vector space over \mathbb{C} , with dimension

$$\mu = \prod_{j=1}^N \left(\frac{1}{q_j} - 1 \right).$$

It is graded by weighted degree, and the elements of top degree form a one-dimensional subspace generated by $\text{hess}(W) = \det \left(\frac{\partial^2 W}{\partial x_i \partial x_j} \right)$. One can check directly that the top degree is equal to \hat{c} .

For $f, g \in \mathcal{Q}_W$, the residue pairing $\langle f, g \rangle$ may be defined by

$$fg = \frac{\langle f, g \rangle}{\mu} \text{hess}(W) + \text{lower order terms.} \quad (4)$$

This pairing is non-degenerate, and endows the Milnor ring with the structure of a Frobenius algebra. For more details, see [AGV].

To define the FJRW ring, we require in addition to W a choice of a group of diagonal symmetries of W . The choice of group heavily affects the resulting structure of the FJRW ring. The maximal group of diagonal symmetries is defined as

$$G_W = \{ (\alpha_1, \alpha_2, \dots, \alpha_N) \subseteq (\mathbb{C}^*)^N \mid W(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_N x_N) = W(x_1, x_2, \dots, x_N) \}$$

Note that G_W always contains the exponential grading element $J = (e^{2\pi i q_1}, e^{2\pi i q_2}, \dots, e^{2\pi i q_N})$. In general, the theory requires that the symmetry group be *admissible* (see [FJR1] section 2.3).

The Landau–Ginzburg Mirror Symmetry Conjecture states the following:

Conjecture (Landau–Ginzburg Mirror Symmetry Conjecture). *For a non-degenerate, quasi-homogeneous, singularity W and diagonal symmetry group G , there is a dual singularity W^T with dual symmetry group G^T so that the FJRW-ring of W/G is isomorphic to an orbifolded Milnor ring of W^T/G^T .*

Remark. (i) We use the notation W^T suggestively for the dual singularity, as we demonstrate in this paper that the Berglund–Hübsch transposed singularity is the appropriate dual in the context of LG via LG mirror symmetry for non-degenerate invertible potentials; (ii) A strengthening of the conjecture applying to Frobenius manifolds should be true. However, a definition of orbifolded Frobenius manifold of W^T/G^T is currently lacking.

We now outline the definition of $\mathcal{H}_{W,G}$ as a \mathbb{C} -vector space, after which we will define the pairing, grading, and multiplication that make $\mathcal{H}_{W,G}$ a Frobenius algebra.

In [FJR1], the state space $\mathcal{H}_{W,G}$ is defined in terms of Lefschetz thimbles:

$$\mathcal{H}_{W,G} = \bigoplus_{\gamma \in G} H^{\text{mid}}(\text{Fix } \gamma, W_\gamma^\infty, \mathcal{Q})^G.$$

For further details, see [FJR1]. For our purposes, it will be most convenient to give a presentation in terms of Milnor rings, but we should point out that the isomorphism between the two presentations is not canonical ([Wa1], [Wa2]).

Let G be an admissible group. For $h \in G$, let $\text{Fix } h \subset \mathbb{C}^N$ be the fixed locus of h , and let N_h be its dimension. Define

$$\mathcal{H}_h := \Omega^{N_h}(\mathbb{C}^{N_h}) / (dW|_{\text{Fix } h} \wedge \Omega^{N_h-1}) \cong \mathcal{Q}_{W|_{\text{Fix } h}} \cdot \omega$$

where $\omega = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{N_h}}$ is a volume form*.

G acts on \mathcal{H}_h via its action on the coordinates, and the state space of the FJRW-ring is the vector space of invariants under this action, i.e.

$$\mathcal{H}_{W,G} := \left(\bigoplus_{h \in G} \mathcal{H}_h \right)^G.$$

$\mathcal{H}_{W,G}$ is \mathbb{Q} -graded by the so-called W -degree, which depends only on the G -grading. To define this grading, first note that each element $h \in G$ can be uniquely expressed as

$$h = (e^{2\pi i \Theta_1^h}, e^{2\pi i \Theta_2^h}, \dots, e^{2\pi i \Theta_N^h})$$

with $0 \leq \Theta_i^h < 1$.

For $\alpha_h \in (\mathcal{H}_h)^G$, the W -degree of α_h is defined by

$$\deg_W(\alpha_h) := \dim \text{Fix } h + 2 \sum_{j=1}^N (\Theta_j^h - q_j). \quad (5)$$

Since $\text{Fix } h = \text{Fix } h^{-1}$, we have $\mathcal{H}_h \cong \mathcal{H}_{h^{-1}}$, and the pairing on $\mathcal{Q}_{W|_{\text{Fix } h}}$ induces a pairing

$$(\mathcal{H}_h)^G \otimes (\mathcal{H}_{h^{-1}})^G \rightarrow \mathbb{C}.$$

The pairing on $\mathcal{H}_{W,G}$ is the direct sum of these pairings. Fixing a basis for $\mathcal{H}_{W,G}$, we denote the pairing by a matrix $\eta_{\alpha,\beta} = \langle \alpha, \beta \rangle$, with inverse $\eta^{\alpha,\beta}$.

For each pair of non-negative integers g and n , with $2g - 2 + n > 0$, the FJRW cohomological field theory produces classes $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \dots, \alpha_N) \in H^*(\overline{\mathcal{M}}_{g,n})$ of complex codimension D for each n -tuple $(\alpha_1, \alpha_2, \dots, \alpha_N) \in (\mathcal{H}_{W,G})^N$. The codimension D is given by

$$D := \hat{c}_W(g-1) + \frac{1}{2} \sum_{i=1}^N \deg_W(\alpha_i),$$

and the n -point correlators are defined to be

$$\langle \alpha_1, \dots, \alpha_N \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n}^W(\alpha_1, \dots, \alpha_N),$$

so $\langle \alpha_1, \dots, \alpha_N \rangle_{g,n}$ obviously vanishes unless the codimension of $\Lambda_{g,n}^W(\alpha_1, \dots, \alpha_N)$ is zero. The ring structure on $\mathcal{H}_{W,G}$ is determined by the genus-zero three-point correlators. In other words, if $r, s \in \mathcal{H}_{W,G}$, then

$$r \star s := \sum_{\alpha, \beta} \langle r, s, \alpha \rangle_{0,3} \eta^{\alpha,\beta} \beta \quad (6)$$

where the sum is taken over all choices of α and β in a fixed basis of $\mathcal{H}_{W,G}$.

The classes $\Lambda_{g,n}^W(\alpha_1, \dots, \alpha_N)$ satisfy the following axioms which facilitate the computation of the three-point correlators $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$.

*Note the volume form encodes a determinant-twist on the natural G -action on $\mathcal{Q}_{W|_{\text{Fix } h}}$.

Axiom 1. Dimension: If the codimension $D \notin \frac{1}{2}\mathbb{Z}$, then $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \dots, \alpha_N) = 0$. In particular, if $g = 0$ and $n = 3$, then $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = 0$ unless $D = 0$, which occurs if and only if $\sum_{i=1}^3 \deg_W \alpha_i = 2\hat{c}$.

Axiom 2. Symmetry: Let $\sigma \in S_N$. Then

$$\langle \alpha_1, \dots, \alpha_N \rangle_{g,n} = \langle \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)} \rangle_{g,n}.$$

The next few axioms relate to the degrees of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_N$ endowing an orbicurve \mathcal{C} with k marked points p_1, \dots, p_k and endowed with a W -structure. This means that for each monomial $\prod_{j=1}^N z_j^{a_{ij}}$ of W , $\bigotimes_{j=1}^N \mathcal{L}_j^{\otimes a_{ij}} \cong \omega_{\log}$. Here, ω_{\log} is the canonical bundle of $\mathcal{C} \setminus \{p_1, \dots, p_k\}$, and the identification of monomials in the \mathcal{L}_j with ω_{\log} arises naturally in the attempt to solve the Witten equation on the orbicurve \mathcal{C} . The details may be found in [FJR1] and provide geometric background to the present construction.

Consider the class $\Lambda_{g,k}^W(\alpha_1, \alpha_2, \dots, \alpha_k)$, with $\alpha_j \in (\mathcal{H}_{h_j})^G$ for each $j \in \{1, \dots, N\}$. For each variable x_j , then $l_j := \deg |\mathcal{L}_j|$ is given by

$$l_j = q_j(2g - 2 + k) - \sum_{i=1}^k \Theta_j^{h_i}. \quad (7)$$

($|\mathcal{L}_j|$ denotes the pushforward of a bundle on the orbicurve \mathcal{C} to the underlying coarse curve).

Axiom 3. Integer degrees: If $l_j \notin \mathbb{Z}$ for some $j \in \{1, \dots, N\}$, then $\Lambda_{g,k}^W(\alpha_1, \alpha_2, \dots, \alpha_k) = 0$.

Remark. This axiom has the following important consequence, which follows immediately from examining Equation (7).

Corollary 2.1. Suppose $\Lambda_{g,k}(\alpha_1, \dots, \alpha_{k-1}, \alpha_k) \neq 0$, with $\alpha_i \in \mathcal{H}_{h_i}$. Then

$$\Lambda_{g,k}(\alpha_1, \dots, \alpha_{k-1}, \tilde{\alpha}_k) = 0 \text{ for any } \tilde{\alpha}_k \notin \mathcal{H}_{h_k}.$$

Proof. Since $\Lambda_{g,k}(\alpha_1, \dots, \alpha_{k-1}, \alpha_k) \neq 0$, we know that for all j

$$l_j = q_j(2g - 2 + k) - \sum_{i=1}^k \Theta_j^{h_i} \in \mathbb{Z}.$$

Suppose $\alpha_k \in \mathcal{H}_{h_k}$, where $\tilde{h}_k = (\tilde{h}_k h_k^{-1})h_k$. In order to have

$$\tilde{l}_j = q_j(2g - 2 + k) - \sum_{i=1}^{k-1} \Theta_j^{h_i} - \Theta_j^{\tilde{h}_k} \in \mathbb{Z},$$

we need $\Theta_j^{\tilde{h}_k h_k^{-1}} \in \mathbb{Z}$.

Now, by Axiom 3, $\Lambda_{g,k}(\alpha_1, \dots, \alpha_{k-1}, \tilde{\alpha}_k) = 0$ unless this holds for all j , which is equivalent to $\tilde{h}_k = h_k$. \square

Axiom 4. Concavity: If $l_j < 0$ for all $j \in \{1, 2, 3\}$, then $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = 1$.

The next axiom is related to the Witten map. When $H^0(\bigoplus_{j=1}^N \mathcal{L}_j)$ and $H^1(\bigoplus_{j=1}^N \mathcal{L}_j)$ have the same rank, the Witten map is given by:

$$\begin{aligned} \mathcal{W} : H^0\left(\bigoplus_{j=1}^N \mathcal{L}_j\right) &\rightarrow H^1\left(\bigoplus_{j=1}^N \mathcal{L}_j\right) \\ \mathcal{W} &= \left(\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \dots, \frac{\partial W}{\partial x_N} \right). \end{aligned}$$

Put $h_j^i = \text{rank } H^i(\mathcal{L}_j)$.

The fact that the Witten map is well-defined is a consequence of the geometric conditions on the \mathcal{L}_j considered in [FJR1]. For further details, we refer readers to the original paper.

If $\Lambda_{g,n}^W(\alpha_1, \dots, \alpha_N)$ is a class of codimension zero, we obtain a complex number by integrating over $\overline{\mathcal{M}}_{g,n}$. Abusing notation, we will refer to the class $\Lambda_{g,n}^W(\alpha_1, \dots, \alpha_N)$ and its integral over $\overline{\mathcal{M}}_{g,n}$ interchangeably.

Axiom 5. *Index-Zero:* Consider the class $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \dots, \alpha_N)$, with $\alpha_i \in \mathcal{H}_{\gamma_i, G}$. If $\text{Fix } \gamma_i = \{0\}$ for each $i \in \{1, 2, \dots, n\}$ and $\Lambda_{g,n}(\alpha_1, \alpha_2, \dots, \alpha_N)$ is of codimension

$$\sum_{j=1}^N (h_j^0 - h_j^1) = 0,$$

then $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \dots, \alpha_N)$ is equal to the degree of the Witten map.

Axiom 6. *Composition:* If the four-point class, $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is of codimension zero, then the correlator $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ decomposes in terms of three-point correlators in the following way:

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle = \sum_{\beta, \delta} \langle \alpha_1, \alpha_2, \beta \rangle \eta^{\beta, \delta} \langle \delta, \alpha_3, \alpha_4 \rangle.$$

Note that $\text{Fix } J = \{0\}$ so $\mathcal{H}_J \cong \mathbb{C}$ and $\deg \mathcal{H}_J = 0$. The identity element in the FJRW-ring is an element of \mathcal{H}_J , and we denote this element by $\mathbf{1}$.

Axiom 7. *Pairing:* For $\alpha_1, \alpha_2 \in \mathcal{H}_{W, G}$, $\langle \alpha_1, \alpha_2, \mathbf{1} \rangle = \eta(\alpha_1, \alpha_2)$, where η is the pairing in $\mathcal{H}_{W, G}$.

Axiom 8. *Sums of singularities:* If $W_1 \in \mathbb{C}[x_1, \dots, x_r]$ and $W_2 \in \mathbb{C}[y_1, \dots, y_s]$ are two non-degenerate, quasi-homogeneous polynomials with maximal symmetry groups G_1 and G_2 , then the maximal symmetry group of $W = W_1 + W_2$ is $G = G_1 \times G_2$, and there is an isomorphism of Frobenius algebras

$$\mathcal{H}_{W, G} \cong \mathcal{H}_{W_1, G_{W_1}} \otimes \mathcal{H}_{W_2, G_{W_2}}$$

Remark. We note an important consequence of Axiom 8. Under the same hypotheses as in the statement of the axiom, we have a Frobenius Algebra isomorphism

$$\mathcal{Q}_W \cong \mathcal{Q}_{W_1} \otimes \mathcal{Q}_{W_2},$$

and similarly

$$\mathcal{Q}_{W^T} \cong \mathcal{Q}_{W_1^T} \otimes \mathcal{Q}_{W_2^T}.$$

Consequently, in order to prove the Mirror Symmetry Conjecture for $W = W_1 + W_2$ a sum of decoupled polynomials (with maximal A -model orbifold group, dual to the trivial B -model orbifold group), it suffices to prove it for W_1 and W_2 individually.

Axiom 9. *Deformation Invariance:* $\Lambda_{g,n}^W(\alpha_1, \alpha_2, \dots, \alpha_N)$ is independent of the representative of W .

2.2. Orbifold B -model. Let $W \in \mathbb{C}[y_1, \dots, y_N]$ be a non-degenerate quasi-homogeneous polynomial where y_i has weight $q_i \in \mathbb{Q}$.

We will take W to be an invertible potential, so $W = \sum W_j$ where each $W_j \in \mathbb{C}[y_1^{(j)}, \dots, y_{n_j}^{(j)}]$ is of loop, chain, or Fermat type.

Let $G \subset (\mathbb{C}^*)^N$ be a group of diagonal symmetries of W .

For $g \in G$, $\text{Fix}(g) = \mathbb{C}^{N_g}$ where $N_g = \dim \text{Fix}(g)$. Put $\mathcal{Q}_g := \mathcal{Q}_{W|_{\text{Fix } g}} \omega_{\text{Fix } g}$, where as before the presence of the volume form $\omega_{\text{Fix } g}$ encodes a determinant twist of the natural G -action of $\mathcal{Q}_{W|_{\text{Fix } g}}$.

Definition 1. The unprojected state space of the Landau–Ginzburg orbifold B -model of W/G is defined to be

$$\mathcal{Q} = \bigoplus_{g \in G} \mathcal{Q}_g.$$

This defines \mathcal{Q} as a G -graded \mathbb{C} -vector space. \mathcal{Q} also possesses a \mathbb{Q} bi-grading, which we discuss in the next section. We will show that the multiplication defined in this section respects the bi-grading.

We endow \mathcal{Q} with a non-degenerate pairing $\langle \cdot, \cdot \rangle$ by taking the sum of the pairings $\mathcal{Q}_g \otimes \mathcal{Q}_{g^{-1}} \rightarrow \mathbb{C}$, which are induced by the residue pairing under the identification $\mathcal{Q}_g \cong \mathcal{Q}_{g^{-1}}$.

We aim to endow \mathcal{Q} with an algebra structure which preserves both the G -grading and the \mathbb{Q} bi-grading. We observe that for $g \in G$, we have a ring homomorphism $\mathcal{Q}_e \rightarrow \mathcal{Q}_g$ given by setting variables not fixed by g equal to zero. This induces on \mathcal{Q}_g the structure of a cyclic \mathcal{Q}_e module, with $1 \in \mathcal{Q}_g$ as the generator of the g -graded summand.

So to define an algebra structure on \mathcal{Q} , it suffices to define a compatible multiplication

$$1_g \star 1_h = \gamma_{g,h} 1_{gh}.$$

Since 1_e will be the identity for the multiplication, we require

$$1_e \star 1_g = 1_g \quad \text{so} \quad \gamma_{e,g} = 1_g = \gamma_{g,e}. \quad (8)$$

For the multiplication to be associative, we must have

$$(1_g \star 1_h) \star 1_k = 1_g \star (1_h \star 1_k) \quad \text{so} \quad \gamma_{g,h} \gamma_{gh,k} = \gamma_{g,hk} \gamma_{h,k}. \quad (9)$$

We propose the following definition of γ and check that it satisfies (8) and (9).

Definition 2. For $g \in G$, let $I_g = \{i : g_i = 1\}$. Define γ through the equation

$$\gamma_{g,h} \frac{\text{hess } W|_{\text{Fix}(g) \cap \text{Fix}(h)}}{\dim \text{Fix}(g) \cap \text{Fix}(h)} = \begin{cases} \frac{\text{hess } W|_{\text{Fix}(gh)}}{\dim \text{Fix}(gh)} & \text{if } I_g \cup I_h \cup I_{gh} = \{1, \dots, n\} \\ 0 & \text{else.} \end{cases} \quad (10)$$

Remark. By definition, $\gamma_{g,h}$ has non-zero pairing with the determinant of the hessian of W on the common fixed locus of g and h , provided each variable is fixed by at least one of g , h and gh . The factor of $\dim \text{Fix}(g) \cap \text{Fix}(h)$ in the denominator ensures that Condition (8) is satisfied.

Proposition 2.2. *The above multiplication \star is associative.*

Proof. This definition obviously satisfies (8), since $1 \in \mathcal{Q}_g$ pairs to unity with $\text{hess } W|_{\text{Fix}(g)}$.

It remains to check the associativity (9) of the putative cocycle γ .

We see here the benefit of restricting our attention to invertible potentials (sums of loops, chains, and Fermat types).

We first check associativity of multiplication when W is of one of these atomic types. The key point here is that if $k \in (\mathbb{C}^*)^N$ is a symmetry of W fixing y_1 , then k acts trivially on all of \mathbb{C}^N . So $1_g \star 1_h = \gamma_{g,h} 1_{gh}$ can be non-zero only if one of g , h , or gh is the identity.

If $g = \text{id}$, $h = \text{id}$, or $k = \text{id}$ then associativity is obvious.

Suppose $g \neq \text{id}$, $h \neq \text{id}$ and $k \neq \text{id}$. We show that both sides of (9) vanish. Consider the left hand side. If $gh \neq \text{id}$ then by the above remark, $1_g \star 1_h = 0$. If $gh = \text{id}$, the left hand side is $\gamma_{g,g^{-1}} 1_k$. Now, $\gamma_{g,g^{-1}}$ pairs with $\text{hess } W|_{\text{Fix}(g)}$, so depends on the variables not fixed by g (in particular y_1). Since $k \neq \text{id}$, y_1 is not fixed by k , and $\gamma_{g,g^{-1}} 1_k = 0 \in \mathcal{Q}_k$. A similar argument applies to the right hand side.

Thus we have an associative multiplication on \mathcal{Q} for W a loop, chain, or Fermat potential. In fact, we have shown furthermore that a triple-product vanishes unless one of the factors is in the identity sector, and the other two factors are in sectors corresponding to mutually inverse group elements.

This multiplication (Definition 2) extends to any invertible potential, as the product may be decomposed into contributions from each atomic summand, and associativity on the summands implies associativity for the whole invertible potential. \square

In the next section, we show that the multiplication on the unprojected state space descends to a multiplication on invariants, without making any assumptions about the potential being of atomic type.

2.2.1. Projecting to invariants. Now we turn our attention to the G -invariants in \mathcal{Q} for the determinant-twisted G action. We make the important restriction that $G \subseteq SL_N \mathbb{C}$, so that the G -invariants in \mathcal{Q}_e are the same whether or not we twist by the determinant on $\text{Fixid} = \mathbb{C}^N$. This means that the \mathcal{Q}_e -module structure on $\mathcal{Q} = \bigoplus_{g \in G} \mathcal{Q}_g$ descends to a $(\mathcal{Q}_e)^G$ -module structure on the determinant-twisted G invariants $\left(\bigoplus_{g \in G} \mathcal{Q}_g\right)^G$. This hypothesis will be justified later when we see that admissible A -model orbifold groups correspond to subgroups of $SL_N \mathbb{C}$ on the B -side.

To see that the product descends to invariants, we prove the following lemma.

Lemma 2.3. *Suppose $H, K \in Q_e$ are monomials such that $H1_h \in \mathcal{Q}_h$ and $K1_k \in \mathcal{Q}_k$ are (determinant-twisted) G -invariants. Then $HK1_h \star 1_k$ is a (determinant-twisted) G -invariant.*

Proof. The lemma is trivially true if $HK1_h \star 1_k = 0$. We may therefore suppose that for each $i \in \{1, \dots, n\}$, at least one of h_i, k_i or $h_i k_i$ equals 1.

G -invariance of the $H1_h$ and $K1_k$ yields

$$g(H) \prod_{i \in I_h} g_i = 1 \quad (11)$$

$$g(K) \prod_{i \in I_k} g_i = 1, \quad (12)$$

where $g(H)$ denotes the phase of the action of g on the monomial H , and similarly for $g(K)$. We need to compute the action of g on $HK1_h \star 1_k$.

Since we assume $1_h \star 1_k \neq 0$, Equation (10) applies. The phase of g on either side of this relation must coincide, so

$$g(\gamma_{h,k}) = \frac{\prod_{i \in I_{hk}} g_i^{-2}}{\prod_{i \in I_h \cap I_k} g_i^{-2}}.$$

Then, using (11) and (12), the phase of g on

$$(H1_h) \star (K1_k) = HK\gamma_{h,k}1_{hk}$$

is

$$\begin{aligned} g(H)g(K)g(\gamma_{h,k}) \prod_{i \in I_{hk}} g_i &= \prod_{i \in I_h} g_i^{-1} \prod_{i \in I_k} g_i^{-1} \frac{\prod_{i \in I_h \cap I_k} g_i^2}{\prod_{i \in I_h \cap I_k} g_i^2} \prod_{i \in I_{hk}} g_i = \prod_{i \in I_h \cup I_k} g_i^{-1} \prod_{i \in I_h \cap I_k} g_i \prod_{i \in I_{hk}} g_i^{-1} \\ &= \prod_{i \in \{1, \dots, n\}} g_i^{-1} = 1 \end{aligned}$$

by the assumption $G \subseteq SL_N \mathbb{C}$.

So the \star -product of G -invariants is again G -invariant. \square

2.2.2. Pairing and Frobenius Algebra. The pairing \langle, \rangle on $\mathcal{Q}_{W,G}$ is the sum of the pairings $\mathcal{Q}_g \otimes \mathcal{Q}_{g^{-1}} \rightarrow \mathbb{C}$, which are induced by the residue pairing under the identification $\mathcal{Q}_g \cong \mathcal{Q}_{g^{-1}}$.

The orbifold Milnor ring (after projecting to G invariants) is a Frobenius Algebra. This follows from the definition of the pairing and the associativity of multiplication.

By construction, the above multiplication preserves the G -grading, and we will show in the next section that it preserves the \mathbb{Q} bi-grading also.

2.3. Bi-grading. Recall the Intriligator–Vafa grading [IV]:

$$J_{\pm} = \pm \sum_{\Theta_i^h \notin \mathbb{Z}} (\Theta_i^h - \frac{1}{2}) + \sum_{\Theta_i^h \in \mathbb{Z}} (q_i - \frac{1}{2}),$$

and

$$\hat{c} = \sum (1 - 2q_i)$$

We introduce the following bi-gradings for Landau–Ginzburg Theories, for a sector corresponding to a symmetry $h = (e^{2\pi i \Theta_1^h}, \dots, e^{2\pi i \Theta_N^h})$ of the potential $W \in \mathbb{C}[X_1, \dots, X_N]$ having charges q_1, \dots, q_N :

| | |
|---------|---|
| Q_+^A | $J_+ + \frac{\hat{c}}{2} = \sum_{\Theta_i^h \notin \mathbb{Z}} (\Theta_i^h - q_i)$ |
| Q_-^A | $-J_- + \frac{\hat{c}}{2} = \sum_{\Theta_i^h \notin \mathbb{Z}} (\Theta_i^h - q_i) + \sum_{\Theta_i^h \in \mathbb{Z}} (1 - 2q_i)$ |

| | |
|---------|--|
| Q_+^B | $J_+ + \frac{\hat{c}}{2} = \sum_{\Theta_i^h \notin \mathbb{Z}} (\Theta_i^h - q_i)$ |
| Q_-^B | $J_- + \frac{\hat{c}}{2} = \sum_{\Theta_i^h \notin \mathbb{Z}} (1 - \Theta_i^h - q_i)$ |

(13)

Remark. The grading above is an ‘external’ grading. The A and B models have ‘internal’ gradings coming from the weighted degree of monomials in the Milnor rings which are summands in the state space. A monomial of weighted degree p has an A -model bi-grading of $(p, -p)$ and a B -model bi-grading of (p, p) .

Note

| | |
|-----------------|--|
| $Q_+^A + Q_-^A$ | $J_+ - J_- + \hat{c} = \dim \text{Fix } h + 2 \sum (\Theta_i^h - q_i)$ |
| $Q_+^A - Q_-^A$ | $J_+ + J_- = \sum_{\Theta_i^h \in \mathbb{Z}} (2q_i - 1)$ |

| | |
|-----------------|--|
| $Q_+^B + Q_-^B$ | $J_+ + J_- + \hat{c} = \sum_{\Theta_i^h \notin \mathbb{Z}} (1 - 2q_i)$ |
| $Q_+^B - Q_-^B$ | $J_+ - J_- = 2 \sum_{\Theta_i^h \notin \mathbb{Z}} (\Theta_i^h - q_i) + \sum_{\Theta_i^h \in \mathbb{Z}} (2q_i - 1)$ |

(14)

So this grading recovers the A -model grading of [FJR1] as the sum of the A -model bi-gradings. The internal A -model bi-grading has following interpretation. $H^N(\mathbb{C}^N, W^\infty, \mathbb{C})$ has a mixed Hodge structure which defines Hodge grading and Hodge decomposition

$$H^N(\mathbb{C}^N, W^\infty, \mathbb{C}) = \bigoplus_p H^{p, N-p}.$$

Under the isomorphism to the Milnor ring, $H^{p, N-p}$ corresponds to the degree p -component of Milnor ring. We have absorbed N into external grading, so the internal bi-grading is $(p, -p)$.

It is desirable to show that the *difference* of the A -model bi-gradings (corrected by twice the internal grading) is preserved under A -model multiplication. In full generality, this has been intractable because it would demand a more precise understanding of the Ramond sector contribution to A -model multiplication than is currently available. However, in the case of maximal A -model symmetry group, the desired fact follows from Theorem 4.1 and the fact that the mirror map of Section 3.2 preserves the bi-grading.

Lemma 2.4. *The B -model multiplication (Def. 2) preserves the bigrading (Q_+^B, Q_-^B) of Eq. (13).*

Proof. To show that the multiplication on the orbifold B -model respects the bi-grading (13), it suffices to show it respects the bi-grading (14), accounting of course for the internal (p, p) grading on the B -model.

Because the B -model is a module over \mathcal{Q}_e , the contribution of the internal grading is obviously additive under multiplication. The contribution of

$$\deg_+^B := \sum_{\Theta_i^h \notin \mathbb{Z}} (1 - 2q_i) \tag{15}$$

is additive under multiplication because if $(1_g \star 1_h) \neq 0$,

$$\begin{aligned} \deg_+^B(1_g \star 1_h) &= Q_+^B + Q_-^B + 2 \deg(\text{hess } W|_{\text{Fix } gh}) - 2 \deg(\text{hess } W|_{\text{Fix}(g) \cap \text{Fix}(h)}) \\ &= \sum_{\Theta_i^{gh} \notin \mathbb{Z}} (1 - 2q_i) + 2 \sum_{\Theta_i^{gh} \in \mathbb{Z}} (1 - 2q_i) - 2 \sum_{\Theta_i^g, \Theta_i^h \in \mathbb{Z}} (1 - 2q_i), \end{aligned}$$

while

$$\deg_+^B(1_g) + \deg_+^B(1_h) = \sum_{\Theta_i^g \notin \mathbb{Z}} (1 - 2q_i) + \sum_{\Theta_i^h \notin \mathbb{Z}} (1 - 2q_i).$$

We check by cases that these are equal (in fact in the two sums, the summand corresponding to the i^{th} variable is the same).

- If $\Theta_i^g \notin \mathbb{Z}$ and $\Theta_i^h \notin \mathbb{Z}$, then $\Theta_i^{gh} \in \mathbb{Z}$ since every variable is fixed by g , h , or gh for non-zero B -model product. So the contribution to the right of each equation is $2(1 - 2q_i)$.
- If $\Theta_i^g \in \mathbb{Z}$ and $\Theta_i^h \notin \mathbb{Z}$ (or vice versa), then $\Theta_i^{gh} \notin \mathbb{Z}$, and the contribution to the right of each equation is $1 - 2q_i$.
- If $\Theta_i^g \in \mathbb{Z}$ and $\Theta_i^h \in \mathbb{Z}$, then the contribution to the right of each equation is 0.

We now check that the B -model multiplication respects

$$\deg_-^B := 2 \sum_{\Theta_i^h \notin \mathbb{Z}} (\Theta_i^h - q_i) + \sum_{\Theta_i^h \in \mathbb{Z}} (2q_i - 1) = \sum_{\Theta_i^h \notin \mathbb{Z}} (2\Theta_i^h - 1). \quad (16)$$

Remark. The internal grading doesn't contribute to \deg_-^B , as the difference of the (p, p) bi-grading is zero).

Now

$$\deg_-^B(1_g \star 1_h) = \sum_{\Theta_i^{gh} \notin \mathbb{Z}} (2\Theta_i^{gh} - 1),$$

and

$$\deg_-^B(1_g) + \deg_-^B(1_h) = \sum_{\Theta_i^g \notin \mathbb{Z}} (2\Theta_i^g - 1) + \sum_{\Theta_i^h \notin \mathbb{Z}} (2\Theta_i^h - 1).$$

Again the two sums match up term by term:

- If $\Theta_i^g \notin \mathbb{Z}$ and $\Theta_i^h \notin \mathbb{Z}$, then $\Theta_i^{gh} \in \mathbb{Z}$ since every variable is fixed by g , h , or gh for non-zero B -model product. Therefore $\Theta_i^g + \Theta_i^h = 1$, and the contribution to the right of each equation is 0.
- If $\Theta_i^g \in \mathbb{Z}$ and $\Theta_i^h \notin \mathbb{Z}$, then $\Theta_i^{gh} = \Theta_i^h \notin \mathbb{Z}$, and the contribution to the right of each equation is $2\Theta_i^h - 1$. Similarly if $\Theta_i^h \in \mathbb{Z}$ and $\Theta_i^g \notin \mathbb{Z}$.
- If $\Theta_i^g \in \mathbb{Z}$ and $\Theta_i^h \in \mathbb{Z}$, then the contribution to the right of each equation is 0.

□

2.4. Relation between A and B model for a fixed singularity. Note that the state spaces of the A and B models for a fixed singularity are isomorphic as vector spaces. For its bi-grading,

$$\deg_+^A = \deg_+^B, \quad \deg_-^A = -\deg_-^B + \hat{c}.$$

This simple relation is particularly relevant in the Calabi-Yau case ($\sum q_i = 1$) where the same relation holds for the Calabi-Yau hypersurface defined by $W = 0$, giving further evidence of Landau-Ginzburg mirror symmetry.

3. MIRROR SYMMETRY FOR STATE SPACES

3.1. Duality of Groups. Following Berglund–Hübsch [BH], we consider the transposed singularity

$$W^T = \sum_{i=1}^N \prod_{j=1}^N Y_j^{a_{ji}},$$

which has exponent matrix A^T .

This suggests writing

$$A^{-1} = \begin{pmatrix} \frac{\bar{\rho}_1}{\bar{\rho}_2} \\ \vdots \\ \frac{\bar{\rho}_1}{\bar{\rho}_N} \end{pmatrix}, \quad \text{with row vectors} \quad \bar{\rho}_i = \begin{bmatrix} \bar{\varphi}_1^{(i)} & \cdots & \bar{\varphi}_N^{(i)} \end{bmatrix}. \quad (17)$$

Comparing to Equation (2), we see $\bar{\varphi}_j^{(i)} = \varphi_i^{(j)}$.

As above, each $\bar{\rho}_k$ is a symmetry of W^T and generate $G_{W^T}^{\max}$, where

$$\bar{\rho}_k Y_j = \exp(2\pi i \bar{\varphi}_j^{(k)}) Y_j,$$

and the exponential grading operator is $\bar{J} = \prod_{i=1}^N \bar{\rho}_i$.

The following lemma is straightforward, but essential to what follows.

Lemma 3.1.

$$\begin{aligned} & \prod_{i=1}^N \rho_i^{\alpha_i} \text{ preserves the monomial } \prod_{j=1}^N X_j^{r_j} \\ \text{if and only if} & \\ & \prod_{j=1}^N \bar{\rho}_j^{r_j} \text{ preserves the monomial } \prod_{i=1}^N Y_i^{\alpha_i}. \end{aligned}$$

Proof. Both statements are equivalent to

$$\begin{bmatrix} r_1, & \cdots, & r_N \end{bmatrix} A_W^{-1} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \in \mathbb{Z}$$

□

In particular, since each $\bar{\rho}_k$ preserves every monomial $\prod_i Y_i^{a_{ij}}$ appearing in W^T , we have that $\prod_i \rho_i^{a_{ij}}$ preserves every X_k . That is:

Corollary 3.2. *Let $W = \sum_{i=1}^N \prod_{j=1}^N X_j^{a_{ij}}$ be a non-degenerate, invertible singularity with exponent matrix $A = (a_{ij})$. Let the symmetry ρ_k be given by the k th column of A^{-1} as above.*

Then

$$\prod_{i=1}^N \rho_i^{a_{ij}} = 1$$

for every $j \in \{1, \dots, n\}$.

Remark. In [K], this observation is attributed to Skarke in the special case of ‘Loop potentials’.

Definition 3. We define dual group G^T as

$$G^T := \left\{ \prod_{i=1}^N \bar{\rho}_i^{r_i} \left| [r_1, \dots, r_N] A_W^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} \in \mathbb{Z} \text{ for all } \prod_{i=1}^N \rho_i^{a_i} \in G \right. \right\}. \quad (18)$$

If $g = \prod_{i=1}^N \rho_i^{a'_i}$ is a different presentation, $\prod_{i=1}^N \rho_i^{a_i - a'_i} = 1$. Hence,

$$A_W^{-1} [a_1 - a'_1, \dots, a_N - a'_N]^T \in \mathbb{Z}^N.$$

Therefore, the above definition is independent of presentation of elements of G .

Lemma 3.3. *Let G be a group of diagonal symmetries of the non-degenerate invertible potential W , and G^T the dual group of symmetries of W^T . Then*

$$(G^T)^T = G.$$

Proof. It is clear from the definition that $G \subseteq (G^T)^T$ and $\mathbb{C}[X_1, \dots, X_N]^G \subseteq \mathbb{C}[X_1, \dots, X_N]^{(G^T)^T}$. This implies that G and $(G^T)^T$ have equal invariant rings, and since the actions on $\mathbb{C}[X_1, \dots, X_N]$ extend to actions on the fraction field with the same fixed field, it follows from field theory that $G = (G^T)^T$. \square

It is also obvious $1^T = G^{\max}$. Now we compute $\langle J \rangle^T$. Since $J = \prod_{i=1}^N \rho_i$, $h = \prod_{i=1}^N \bar{\rho}_i^{r_i} \in \langle J \rangle^T$ if and only if $\sum_i r_i q_i \in \mathbb{Z}$. Since $\sum_i r_i q_i$ is precisely the phase of $\det(h)$, we have $\langle J \rangle^T = SL_N \mathbb{C} \cap G_{W^T}^{\max}$.

This explains the SL restriction made in the proof of Lemma 2.3 that the orbifold B -model multiplication descends to the invariants under the action of the orbifold group.

We can use the argument from the proof of 3.3 to settle a question suggested in [FJR1], namely whether any diagonal symmetry group containing J satisfies the following definition of admissible groups.

Definition 4 ([FJR1] Defn 2.3.2). We say that a subgroup $G \leq G_W^{\max}$ is *admissible* or is an *admissible group of Abelian symmetries of W* if there exists a Laurent polynomial Z , quasi-homogeneous with the same weights q_i as W , but with no monomials in common with W , and such that $G = G_{W+Z}$.

Proposition 3.4. *For $W \in \mathbb{C}[X_1, \dots, X_N]$ a non-degenerate (not-necessarily invertible) potential, any group of diagonal symmetries of W containing J is admissible.*

Proof. For a group G of diagonal symmetries of W containing J to be admissible, we require the existence of a Laurent polynomial Z in X_1, \dots, X_N , quasi-homogeneous with the same weights as W , such that G is the *maximal* diagonal symmetry group of $W + Z$.

Now, the ring of G -invariants is finitely generated by monomials. If we let Z be the sum of those G -invariant monomials not divisible by monomials in W , G is the maximal diagonal symmetry group of $W + Z$. (Otherwise there is a diagonal symmetry group H , with $G \subseteq H$ and $\mathbb{C}[X_1, \dots, X_N]^G \subseteq \mathbb{C}[X_1, \dots, X_N]^H$, implying $G = H$ as before). Since J preserves each of the constituent monomials of Z , each of these monomials has integral quasi-homogeneous degree. We may correct each of these monomials by a (negative) power of any monomial in W to ensure that each of the monomials has quasi-homogeneous degree equal to 1, and since we are correcting by G -invariants not dividing the monomials of Z , we do not change the maximal symmetry group of $W + Z$. \square

3.2. Mirror Map. We propose in this section a ‘Mirror map’ $\mathcal{Q}_{W^T, G^T} \rightarrow \mathcal{H}_{W, G}$.

Definition 5 (Mirror Map). Let W be a non-degenerate, invertible potential and G and admissible A -model diagonal symmetry group of W . Define the linear map $\mathcal{Q}_{W^T, G^T} \rightarrow \mathcal{H}_{W, G}$ by the following map on generators

$$\left(\bigoplus_{g \in G^T} \mathcal{Q}_g^{W^T} \right)^{G^T} \longrightarrow \left(\bigoplus_{g \in G} \mathcal{H}_g^W \right)^G \quad (19)$$

$$\prod Y_j^{\alpha_j} dY_j \mid \prod \bar{\rho}_j^{r_j+1} \rangle \mapsto \prod X_j^{r_j} dX_j \mid \prod \rho_j^{\alpha_j+1} \rangle,$$

where it should be understood that the range of the product over X 's is the same as the range of the product over the $\bar{\rho}$'s, and similarly for the Y 's and the ρ 's.

Remark. One may worry that this map is not well-defined, since the monomial $\prod X_j^{r_j} dX_j$ may not be uniquely determined by an element of G^T . However, this monomial is completely determined given both the group element *and* the locus over which to extend the product, namely the fixed locus of the element of G corresponding to $\prod Y_j^{\alpha_j} dY_j$.

We shall see how this correspondence arises in the proof of Theorem 3.5.

Example. We present here the example of the two-variable loop potential $W = x^3y + xy^5$, orbifolded by $J = (e^{(2\pi i)2/7}, e^{(2\pi i)1/7})$ on the A -side and by the dual group $J^T = \langle (-1, -1) \rangle$ on the B -side. The table below presents the vector space generators for the A -model W/J and the B -model W^T/J^T , along with the bigrading. We denote the standard volume form on $\text{Fix } \rho_x^a \rho_y^b$ by $e_{\rho_x^a \rho_y^b}$. The A and B model invariants in each column correspond to each other under the Mirror Map (Equation (19)), and evidently the bi-grading is preserved.

| W/J | $e_{\rho_x^1 \rho_y^1}$ | $e_{\rho_x^2 \rho_y^2}$ | $e_{\rho_x^0 \rho_y^2}$ | $e_{\rho_x^1 \rho_y^3}$ | $e_{\rho_x^2 \rho_y^4}$ | $e_{\rho_x^0 \rho_y^4}$ | $x^2 e_{\rho_x^0 \rho_y^0}$ | $xy e_{\rho_x^0 \rho_y^0}$ | $y^4 e_{\rho_x^0 \rho_y^0}$ |
|------------------|-------------------------------------|--|---|---|--|---|---|-------------------------------------|---|
| deg_+^A | 0 | $\frac{6}{7}$ | $\frac{12}{7}$ | $\frac{4}{7}$ | $\frac{10}{7}$ | $\frac{16}{7}$ | $\frac{8}{7}$ | $\frac{8}{7}$ | $\frac{8}{7}$ |
| deg_-^A | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| W^T/SL | $e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $xy e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $x^2 y^2 e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $y^2 e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $xy^3 e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $y^4 e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $x^2 e_{\bar{\rho}_x^3 \bar{\rho}_y^1}$ | $e_{\bar{\rho}_x^2 \bar{\rho}_y^3}$ | $y^4 e_{\bar{\rho}_x^1 \bar{\rho}_y^5}$ |
| deg_+^B | 0 | $\frac{6}{7}$ | $\frac{12}{7}$ | $\frac{4}{7}$ | $\frac{10}{7}$ | $\frac{16}{7}$ | $\frac{8}{7}$ | $\frac{8}{7}$ | $\frac{8}{7}$ |
| deg_-^B | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Example. Now we present the example of the two-variable chain potential $W = x^3y + y^4$, orbifolded by $J = (e^{2\pi i/4}, e^{2\pi i/4})$ on the A -side and by the dual group $J^T = \langle (e^{2\pi i/3}, e^{2\pi i/6}) \rangle$ on the B -side. The table below presents the vector space generators for the A -model W/J and the B -model W^T/J^T , along with the bigrading. The A and B model invariants in each column correspond to each other under the Mirror Map (Equation (19)), and we see again that the bi-grading is preserved.

| W/J | $e_{\rho_x^1 \rho_y^1}$ | $e_{\rho_x^2 \rho_y^2}$ | $e_{\rho_x^0 \rho_y^3}$ | $x^2 e_{\rho_x^0 \rho_y^0}$ | $xy e_{\rho_x^0 \rho_y^0}$ | $y^2 e_{\rho_x^0 \rho_y^0}$ |
|------------------|-------------------------------------|--|---|-------------------------------------|-------------------------------------|-------------------------------------|
| deg_+^A | 0 | 1 | 2 | 1 | 1 | 1 |
| deg_-^A | 0 | 0 | 0 | 0 | 0 | 0 |
| W^T/SL | $e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $xy e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $x^2 y^2 e_{\bar{\rho}_x^0 \bar{\rho}_y^0}$ | $e_{\bar{\rho}_x^3 \bar{\rho}_y^1}$ | $e_{\bar{\rho}_x^2 \bar{\rho}_y^3}$ | $e_{\bar{\rho}_x^1 \bar{\rho}_y^5}$ |
| deg_+^B | 0 | 1 | 2 | 0 | 1 | 1 |
| deg_-^B | 0 | 0 | 0 | 0 | 0 | 0 |

3.3. Mirror Symmetry for State Spaces. In this section, we prove that the Mirror Map (Equation (19)) is a bi-degree preserving vector space isomorphism.

Theorem 3.5. *Let W be a non-degenerate invertible potential and G an admissible group of diagonal symmetries of W . The Mirror Map defined on generators by Equation (19) is a bi-degree preserving isomorphism of vector spaces.*

Remark. For $G = G_W^{\max}$ and ignoring the bi-grading, this recovers the main result of [K].

The proof proceeds in two steps. First, we consider the *total unprojected mirror map* (Definition 6), which we prove to be a bi-degree preserving vector space isomorphism.

Second, we note that by definition of the dual group, this restricts to an isomorphism on invariants (under the G -action for the A model and the G^T action for the B -model).

Definition 6 (Total Unprojected Mirror Map). Let W be a non-degenerate, invertible potential and G and admissible A -model diagonal symmetry group of W . Define the linear map $\mathcal{Q}_{W^T, G^T} \rightarrow \mathcal{H}_{W, G}$ by the following map on generators

$$\bigoplus_{g \in G_{W^T}^{\max}} \mathcal{Q}_g^{W^T} \longrightarrow \bigoplus_{g \in G_W^{\max}} \mathcal{H}_g^W \quad (20)$$

$$\prod Y_j^{\alpha_j} dY_j \left| \prod \bar{\rho}_j^{r_j+1} \right\rangle \mapsto \prod X_j^{r_j} dX_j \left| \prod \rho_j^{\alpha_j+1} \right\rangle,$$

where the ambiguity in the range of the products is resolved as in Definition 5.

Theorem 3.6. *Let W be a non-degenerate invertible potential. The Total Unprojected Mirror Map defined on generators by Equation (20) is a bi-degree preserving isomorphism of vector spaces.*

Proof. Note that to show the total unprojected mirror map (Equation (20)) is a degree-preserving isomorphism of vector spaces, it suffices to do so for all invertible potentials. To see this, suppose $W = \sum_i W_i$ is a sum of atomic invertible potentials, with $G_W^{\max} = \bigoplus_i G_{W_i}^{\max}$, and similar decompositions for W^T . Suppose the total unprojected mirror map prescribes

$$\prod_i H_i \left| \bigoplus_i h_i \right\rangle \mapsto \prod_i G_i \left| \bigoplus_i g_i \right\rangle,$$

where $g_i \in G_{W_i}^{\max}$ and $G_i \in \mathcal{Q}_{W_i | \text{Fix } g_i}$ is a monomial, and similarly for the h_i and H_i . For the mirror map to be an isomorphism, we require that in the A -model sector $\left| \bigoplus g_i \right\rangle$ corresponding to the B -model monomial $\prod_i H_i$, there is a unique monomial $\prod_i G_i$ which corresponds to $\left| \bigoplus h_i \right\rangle$. This is clearly equivalent to the same holding for each atomic potential W_i .

Inspection of Equations (13) and (14) indicates that the bi-degrees are simply sums of contributions from each atomic summand, so if the total unprojected mirror map preserves bidegree for atomic potentials, it does so for all invertible potentials.

We may therefore restrict our attention to the invertible potentials of Fermat, Loop and Chain type, as the result follows for all invertible potentials from these atomic cases. For each of these cases, we will prove that Equation (20) is a bi-degree preserving vector space isomorphism.

By (14), the sum of the A -model bi-gradings is the A -model degree of the h -twisted sector given in [FJR1]. For this reason, we will show that the bi-grading is preserved under the Mirror Map by showing that the sum and difference of the bi-degrees is preserved.

3.3.1. *Fermat:* $W = X^N$. The total unprojected mirror map is defined on generators by:

$$Y^k dY \left| \text{id} \right\rangle \mapsto 1 \left| \rho^{k+1} \right\rangle, \quad 0 \leq k < N-1,$$

and

$$1 \left| \rho^{k+1} \right\rangle \mapsto X^k dX \left| \text{id} \right\rangle, \quad 0 \leq k < N-1,$$

which evidently yields an isomorphism of unprojected state spaces.

To see that this isomorphism preserves bi-degree, note that for

$$\left| \rho^{k+1} \right\rangle, \quad 0 \leq k < N-1,$$

we have

$$Q_+^B + Q_-^B + 2 \deg_{\text{internal}}^B = 2k\bar{q},$$

and

$$Q_+^A + Q_-^A = 2((k+1)q - q) = 2k\bar{q}.$$

Also, it is clear that

$$Q_+^B - Q_-^B = 0 = Q_+^A - Q_-^A + 2 \deg_{\text{internal}}^A.$$

For

$$1 | \bar{p}^{k+1} \rangle \mapsto X^k dX | \text{id} \rangle, \quad 0 \leq k < N-1,$$

we have

$$Q_+^B + Q_-^B + 2 \deg_{\text{internal}}^B = 1 - 2\bar{q} = Q_+^A + Q_-^A,$$

and

$$Q_+^B - Q_-^B = (2q - 1) + 2kq = Q_+^A - Q_-^A + 2 \deg_{\text{internal}}^A.$$

3.3.2. *Loop*: $W = \sum_{i=1}^N X_i^{a_i} X_{i+1}$. (The subscripts are taken modulo N).

The structure of the loop potential means that the only group element with non-trivial fixed locus is the identity. Therefore we study the total unprojected mirror map out of the B -model identity sector and twisted sectors separately.

Identity B-model sector:

$$\prod_{j=1}^N Y_j^{\alpha_j} dY_j | \text{id} \rangle \mapsto \prod X_j^{r_j} dX_j \left| \prod_{j=1}^N \rho_j^{\alpha_j+1} \right\rangle,$$

where we are purposefully vague about the range of the product for the A -model monomial, since it may either be empty (in which case the monomial should be interpreted as 1) or it may run from 1 to N (when the B -model monomial corresponds to the A -model identity group element).

In the first case, we have

$$\prod_{j=1}^N Y_j^{\alpha_j} dY_j | \text{id} \rangle \mapsto 1 \left| \prod_{j=1}^N \rho_j^{\alpha_j+1} \right\rangle,$$

so

$$\begin{aligned} Q_+^A + Q_-^A &= \dim \text{Fix} + 2 \sum_{j=1}^N (\Theta_j - q_j) \\ &= 2 \sum_{j=1}^N \left(\sum_{i=1}^N (\alpha_i + 1) \varphi_j^{(i)} - \sum_{i=1}^N \varphi_j^{(i)} \right) = 2 \sum_{i=1}^N \alpha_i \left(\sum_{j=1}^N \varphi_j^{(i)} \right) = 2 \sum_{i=1}^N \alpha_i \bar{q}_i \\ &= Q_+^B + Q_-^B + 2 \deg_{\text{internal}}^B \end{aligned}$$

and

$$Q_+^B - Q_-^B = 0 = Q_+^A - Q_-^A + 2 \deg_{\text{internal}}^A$$

On the other hand, if $\prod_{j=1}^N \rho_j^{\alpha_j+1} = \text{id}$, the mirror map looks like

$$\prod_{j=1}^N Y_j^{\alpha_j} dY_j \left| \text{id} = \prod_{j=1}^N \bar{\rho}_j^{r_j+1} \right\rangle \mapsto \prod_{j=1}^N X_j^{r_j} dX_j \left| \prod_{j=1}^N \rho_j^{\alpha_j+1} \right\rangle,$$

and we by Lemma 1.6 we must have N even and α_j, r_j both alternately $a_j - 1$ and 0.

Remark. Evidently, there is a choice of ‘parity’ in the mirror map. For the purpose of establishing an isomorphism of graded vector-spaces, let us suppose that the monomials in the X ’s and the Y ’s have the same parity.

Then,

$$Q_+^B + Q_-^B + 2 \deg_{\text{internal}}^B = 2 \sum_{j=1}^N \alpha_j \bar{q}_j,$$

and

$$\begin{aligned} Q_+^A + Q_-^A &= \dim \text{Fix} \left(\prod_{j=1}^N \rho_j^{\alpha_j+1} \right) + 2 \sum_{j=1}^N (\Theta_j - q_j) = \dots \\ &= \sum_{j=1}^N (1 - 2\bar{q}_j) = 2 \sum_{\substack{j=1 \\ j \text{ odd/even}}}^N (1 - \bar{q}_{j-1} - \bar{q}_j) = 2 \sum_{\substack{j=1 \\ j \text{ odd/even}}}^N (a_j - 1) \bar{q}_j = 2 \sum_{j=1}^N \alpha_j \bar{q}_j. \end{aligned}$$

Again, it is evident that

$$Q_+^B - Q_-^B = 0,$$

while

$$Q_+^A - Q_-^A + 2 \deg_{\text{internal}}^A = \sum_{j=1}^N (2q_i - 1) + 2 \sum_{j=1}^N r_j q_j = 0,$$

by a calculation identical to the one in the preceding paragraph.

Twisted B-model sectors:

Since the B -model twisted sectors have trivial fixed loci, the mirror map sends them all to the A -model untwisted sector.

$$1 \left| \prod_{j=1}^N \bar{\rho}_j^{r_j+1} \right\rangle \mapsto \prod_{j=1}^N X_j^{r_j} dX_j \mid \text{id} \rangle,$$

Hence it is easy to see that

$$Q_+^B + Q_-^B + 2 \deg_{\text{internal}}^B = \hat{c} = Q_+^A + Q_-^A.$$

Furthermore,

$$Q_+^B - Q_-^B = 2 \sum_{j=1}^N r_j q_j - \hat{c} = Q_+^A - Q_-^A + 2 \deg_{\text{internal}}^A$$

follows after applying our computation of \deg_+^A to the computation of \deg_-^B .

It is clear from the cases considered above that the total unprojected mirror map described above yields an isomorphism of vector spaces.

3.3.3. Chain: $W = X_1^{a_1} X_2 + X_2^{a_2} X_3 + \dots + X_{N-1}^{a_{N-1}} X_N + X_N^{a_N}$. This case is more involved than the others, because a symmetry of the chain potential may fix $\{X_s, X_{s+1}, \dots, X_N\}$ for any $s = 1, \dots, N$ or it may have trivial fixed locus.

The total mirror map acts on generators via

$$\prod_{j=1}^t Y_j^{\alpha_j} dY_j \left| \prod_{j=s}^N \bar{\rho}_j^{r_j+1} \right\rangle \mapsto \prod_{j=s}^N X_j^{r_j} dX_j \left| \prod_{j=1}^t \rho_j^{\alpha_j+1} \right\rangle,$$

where $\{Y_1, \dots, Y_t\}$ are the B -model fixed variables and $\{X_s, \dots, X_N\}$ are the A -model fixed variables. We will consider $t = 0$ and $s = N + 1$ to denote trivial fixed loci, and empty products and sums will be assumed to equal 1 and 0 respectively.

Remark. Although the group elements determine these loci, the presentation of these elements as corresponding to specific monomials is not canonical.

For the A -model fixed locus to be $\{X_s, \dots, X_N\}$, we must have $s \leq t+1$ and $\alpha_j = \delta_{t-j}^{\text{even}}(a_j - 1)$ for $s \leq j \leq t$. Further, since $Y_{s-1} \prod_{j=s}^t Y_j^{\delta_{t-j}^{\text{even}}(a_j - 1)}$ vanishes in $\mathcal{Q}_{W^T|_{\{Y_1, \dots, Y_t\}}}$ when $t-s$ is even, we see that $t-s$ must be odd. *i.e.* There is an even number of elements (possibly zero) in $\{s, s+1, \dots, t\}$.

This will allow us to exploit the Remark following Lemma 1.5, in which we noted that the natural map from B -model Milnor ring elements $\prod_i Y_i^{\alpha_i} dY_i$ to A -model symmetries $\prod_i \rho_i^{\alpha_i} J$ maps injectively onto the collection of symmetries with even-dimensional fixed locus.

To see that the total mirror map is a bijection for the chain potential W , suppose $\prod_{j=s}^N X_j^{r_j} dX_j$ is an A -model monomial in the target sector $|g\rangle$, where g fixes the variables $\{X_s, \dots, X_N\}$. To prove bijectivity, we must find in the B -model sector $|h\rangle = \left| \prod_{j=s}^N \bar{\rho}_j^{r_j+1} \right\rangle$ with fixed variables $\{1, \dots, t\}$ a unique monomial $\prod_{j=1}^t Y_j^{\alpha_j} dY_j$ such that $\left| \prod_{j=1}^t \rho_j^{\alpha_j+1} \right\rangle = |g\rangle$. This can be done because for the chain $W^T|_{\{Y_1, \dots, Y_t\}}$, we have $\mathcal{Q}_{W^T|_{\{Y_1, \dots, Y_N\}}}$ mapping injectively onto the collection of A -model symmetries fixing an even number of variables $\{X_s, \dots, X_t\}$. Consequently g corresponds to a unique $\prod_{i=1}^t Y_i^{\alpha_i} dY_i \in \mathcal{Q}_{W^T|_{\text{Fix } h}}$ and the map is bijective as claimed.

We now proceed to compare the bi-gradings on either side of the total mirror map.

$$\begin{aligned}
Q_+^A + Q_-^A &= \dim \text{Fix}(h) + 2 \sum_{j=1}^N (\Theta_j - q_j) \\
&= 2 \left(\sum_{j=1}^t \alpha_j \bar{q}_j \right) + 2 \left(\sum_{\substack{j=t+1 \\ N-j \text{ even}}}^N (a_j - 1) \bar{q}_j \right) + 2\delta_{N-(t+1)}^{\text{even}} \bar{q}_t - \delta_{N-s}^{\text{even}} \\
&= 2 \left(\sum_{j=1}^t \alpha_j \bar{q}_j \right) + 2 \left(\sum_{\substack{j=t+1 \\ N-j \text{ even}}}^N (1 - \bar{q}_j - \bar{q}_{j-1}) \right) + 2\delta_{N-(t+1)}^{\text{even}} \bar{q}_t - \delta_{N-s}^{\text{even}} \\
&= 2 \left(\sum_{j=1}^t \alpha_j \bar{q}_j \right) + \left(\sum_{j=t+1}^N (1 - 2\bar{q}_j) \right) + \delta_{N-(t+1)}^{\text{even}} - \delta_{N-s}^{\text{even}} \\
&= 2 \left(\sum_{j=1}^t \alpha_j \bar{q}_j \right) + \left(\sum_{j=t+1}^N (1 - 2\bar{q}_j) \right) \\
&= Q_+^B + Q_-^B + 2 \deg_{\text{internal}}^B,
\end{aligned}$$

since we have observed that s and $t+1$ have the same parity.

Now consider

$$\begin{aligned}
Q_+^B - Q_-^B &= 2 \sum_{j=t+1}^N (\Theta_j - \bar{q}_j) + \sum_{j=t+1}^N (2\bar{q}_j - 1) \\
&= t + 2 \sum_{j=1}^N (\Theta_j - \bar{q}_j) + \sum_{j=1}^N (2\bar{q}_j - 1) \\
&= \sum_{j=1}^t (1 - 2q_j) + 2 \left(\sum_{j=t+1}^N r_j q_j \right) + \sum_{j=1}^N (2q_j - 1) \\
&= \sum_{j=s}^{t-1} (1 - 2q_j) + 2 \left(\sum_{j=t+1}^N r_j q_j \right) + \sum_{j=s}^N (2q_j - 1)
\end{aligned}$$

$$\begin{aligned}
&= 2 \sum_{j=s}^{t-1} \delta_{t-1-j}^{\text{even}} (a_j - 1) + 2 \left(\sum_{j=t+1}^N r_j q_j \right) + \sum_{j=s}^N (2q_j - 1) \\
&= 2 \sum_{j=s}^N r_j q_j + \sum_{j=s}^N (2q_j - 1) \\
&= Q_+^A - Q_-^A + 2 \deg_{\text{internal}}^A,
\end{aligned}$$

where we could change to q 's from \bar{q} 's because $\sum_j q_j = (1, \dots, 1)A^{-1}(1, \dots, 1)^T = \sum_j \bar{q}_j$. \square

As already indicated, to complete the proof of Theorem 3.5, we observe simply that it is obtained by restricting the total unprojected mirror isomorphism of Theorem 3.6 to \mathcal{Q}_{W^T, G^T} to obtain an isomorphism onto its image. By definition of the Dual Group, this image is $\mathcal{H}_{W, G}$.

4. MIRROR SYMMETRY FOR FROBENIUS ALGEBRAS

4.1. Maximal Symmetry Group. We prove the following theorem,

Theorem 4.1. *Let $W: \mathbb{C}^N \rightarrow \mathbb{C}$ be a non-degenerate, invertible potential with maximal diagonal symmetry group G^{\max} and all charges $q_j < \frac{1}{2}$. Let W^T be the Berglund–Hübsch dual singularity of W , with Milnor ring \mathcal{Q}_{W^T} . Then*

$$\mathcal{H}_{W, G^{\max}} \cong \mathcal{Q}_{W^T}$$

as Frobenius algebra, i.e. The maximally orbifolded A -model ring of W is isomorphic to the unorbifolded B -model ring of W^T .

The restriction to $q_j < \frac{1}{2}$ ensures that the ring generators of $\mathcal{H}_{W, G^{\max}}$ are in Neveu-Schwartz sectors, for which the FJRW multiplication can be computed using algebro-geometric methods.

Note this corresponds to the duality of state spaces, since G^{\max} is dual to the trivial group. However, the linear isomorphism in Theorem 4.1 may in general differ from that of Theorem 3.5. In the earlier theorem, there was a choice of parity involved in the presentation of the Mirror Map for loop potentials. We have been unable to determine whether this choice is compatible with the FJRW product structure on the A -model.

Remark. We would like to note that in the case $N = 2$, Theorem 4.1 has been proven independently by Fan-Shen [FS] in the case of chain potentials, and Acosta [A] in the case of loop potentials. The Fan-Shen result applies more generally to two-variable chains with *any* admissible A -model symmetry group.

Notation. To make the notation less cumbersome in this case, we will omit the group notation for the B -model sector (although the reader should recall that any B -model monomial is implicitly followed by $dY_1 \wedge \dots \wedge dY_N$).

To prove the theorem, we recall that by combining the remark following Axiom 8 and the classification of invertible potentials ([KS], recalled in Section 1.3), it suffices to prove Theorem (4.1) for singularities of Fermat, Loop and Chain type, which we address individually below.

4.1.1. Fermat Potentials: $W = X^a$. The Mirror Theorem in this case was proved as the A_r case of the ‘self-duality’ theorem in [FJR1]. The essential point here is that the exponent matrix is equal to its transpose in the self-dual cases proved in [FJR1]. We show here that the self-duality is in a sense coincidental, and that in general it is the transposed singularity W^T which is mirror to W on the B -model side.

4.1.2. *Loop Potentials:* $W = \sum_{i=1}^N X_i^{a_i} X_{i+1}$. Since degree is additive under multiplication in \mathcal{Q}_{W^T} and in $\mathcal{H}_W^{G^{\max}}$, the isomorphism (19) of graded vector spaces suggests that the desired ring isomorphism

$$\mathcal{Q}_{W^T} \xrightarrow{\cong} \mathcal{H}_{W, G^{\max}} \quad (21)$$

should be induced by the map

$$\begin{aligned} \mathbb{C}[Y_1, \dots, Y_N] &\longrightarrow \mathcal{H}_{W, G^{\max}} \\ Y_i &\longmapsto 1_{\rho_i J}, \end{aligned} \quad (22)$$

where for $g \in G^{\max}$, 1_g denotes the identity in $\mathcal{H}_g^{G^{\max}} \cong \mathcal{Q}_{W|_{\text{Fix}(g)}}^{G^{\max}}$, and the map is extended to $\mathbb{C}[Y_1, \dots, Y_N]$ by multiplicativity.

The following two lemmas show that $\mathcal{H}_{W, G^{\max}}$ is generated by the elements $1_{\rho_i J}$, subject to the relations

$$(1_{\rho_k J})^{\star a_k} + a_{k-1} 1_{\rho_{k-2} J} \star (1_{\rho_{k-1} J})^{\star(a_{k-1}-1)} = 0.$$

This means that the kernel of the above map is precisely the Jacobian ideal dW^T , yielding the desired isomorphism.

We proceed to prove the necessary lemmas.

Notation. Define

$$\rho^\alpha := \prod_{i=1}^N \rho_i^{\alpha_i}.$$

Lemma 4.2. *If $\alpha_i + \beta_i \leq a_i - 1$ for $i \in \{1, \dots, N\}$, and $\text{id} \notin \{\rho^\alpha J, \rho^\beta J, \rho^{\alpha+\beta} J\}$, then*

$$1_{\rho^\alpha J} \star 1_{\rho^\beta J} = 1_{\rho^{(\alpha+\beta)} J}.$$

Proof. The lemma is obviously true when $\rho^\alpha = \text{id}$ or $\rho^\beta = \text{id}$, since 1_J is the multiplicative identity in $\mathcal{H}_W^{G^{\max}}$.

By definition (Equation (6)),

$$1_{\rho^\alpha J} \star 1_{\rho^\beta J} = \sum_{\mu, \nu} \langle 1_{\rho^\alpha J}, 1_{\rho^\beta J}, \mu \rangle \eta^{\mu\nu} \nu.$$

For the three point correlator $\langle 1_{\rho^\alpha J}, 1_{\rho^\beta J}, \mu \rangle$ to be non-zero, we must have

$$\deg 1_{\rho^\alpha J} + \deg 1_{\rho^\beta J} + \deg \mu = 2\hat{c}.$$

By Corollary 2.1, $\mu \in \mathcal{H}_{gJ}$ for the unique $g = \rho^\gamma \in G^{\max}$ satisfying the condition

$$\sum_{i=1}^N (\alpha_i + \beta_i + \gamma_i) \bar{q}_i = 2 \sum_{i=1}^N (a_i - 1) \bar{q}_i$$

and having line bundles $|\mathcal{L}_j|$ of integral degree.

Note that since $\sum_i q_i = \sum_i \bar{q}_i$ and $a_i q_i + q_{i+1} = 1$ we have

$$2\hat{c} = 2 \sum_{i=1}^N (1 - 2q_i) = 2 \sum_{i=1}^N (a_i q_i - q_i) = 2 \sum_{i=1}^N (a_i - 1) \bar{q}_i = \deg 1_{\rho^{\max} J}.$$

Since $0 \leq \alpha_i + \beta_i \leq a_i - 1$ by hypothesis, $\gamma_i = a_i - 1 - \alpha_i - \beta_i$ potentially prescribes the group element g , and we demonstrate below that the corresponding line bundles indeed have integral degree.

We compute the degrees l_j of the line bundles $|\mathcal{L}_j|$, using the formula

$$l_j = q_j(2g - 2 + k) - \sum_{i=1}^k \Theta_j^{h_i},$$

Where g is the genus of the correlator (zero in this case), k is the number of insertions (i.e. three), $h_i \in G^{\max}$ is the group grading of the i^{th} insertion, and $\Theta_j^{h_i}$ is the phase of the action of h_i on X_j .

$$\begin{aligned} l_j &= q_j - (\Theta_j^{\rho^\alpha J} + \Theta_j^{\rho^\beta J} + \Theta_j^{\rho^\gamma J}) = q_j - \sum_{i=1}^N (\alpha_i + 1) \varphi_j^{(i)} - \sum_{i=1}^N (\beta_i + 1) \varphi_j^{(i)} - \sum_{i=1}^N (\gamma_i + 1) \varphi_j^{(i)} \\ &= -2q_j - \sum_{i=1}^N (\alpha_i + \beta_i + \gamma_i) \varphi_j^{(i)} = -2q_j - \sum_{i=1}^N (a_i - 1) \varphi_j^{(i)} = -2q_j - \sum_{i=1}^N (\delta_{i,j} - \varphi_j^{(i-1)} - \varphi_j^{(i)}) = -1 \end{aligned}$$

By the concavity axiom (Axiom 4), $\langle 1_{\rho^\alpha J}, 1_{\rho^\beta J}, \mu \rangle = 1$. Since μ and ν correspond to sectors with trivial fixed loci, $\eta^{\mu\nu} = 1$.

We conclude on substituting into Equation (4.1.2) that

$$1_{\rho^\alpha J} \star 1_{\rho^\beta J} = 1_{\rho^{\alpha+\beta} J},$$

as claimed. \square

Lemma 4.3. *For $N > 2$,*

$$1_{\rho_k J}^{a_k} = -a_{k-1} 1_{\rho_{k-2} \rho_{k-1}^{a_{k-1}-1} J}.$$

Proof. By definition,

$$1_{\rho_k^{(a_k-1)} J} \star 1_{\rho_k J} = \sum_{\mu, \nu} \langle 1_{\rho_k^{(a_k-1)} J}, 1_{\rho_k J}, \mu \rangle \eta^{\mu\nu} \nu. \quad (23)$$

The only non-zero term in the sum occurs when

$$\deg 1_{\rho_k^{(a_k-1)} J} + \deg 1_{\rho_k J} + \deg \mu = 2\hat{c}$$

and the corresponding line bundles have integral degree.

This first condition is equivalent to

$$(a_k - 1) \bar{q}_k + \bar{q}_k + \deg \mu = \sum_{i=1}^N (a_i - 1) \bar{q}_i.$$

Recalling that for all i , $a_i \bar{q}_i + \bar{q}_{i-1} = 1$, so

$$\bar{q}_{k-2} + (a_{k-1} - 1) \bar{q}_{k-1} = 1 - \bar{q}_{k-1} = a_k \bar{q}_k,$$

and denoting $g = \rho^\gamma$, we see that

$$\sum_{i=1}^N \gamma_i \bar{q}_i = \sum_{i=1}^N (a_i - 1) \bar{q}_i - (a_k - 1) \bar{q}_k - \bar{q}_k = \sum_{i=1}^N (a_i - 1) \bar{q}_i - \bar{q}_{k-2} - (a_{k-1} - 1) \bar{q}_{k-1}.$$

Thus we can solve for γ_i in the range $0 \leq \gamma_i < a_i$, namely

$$\gamma_i = (a_i - 1) - \delta_{i,k-1} (a_{k-1} - 1) - \delta_{i,k-2}.$$

We now confirm that the line-bundles which determine the correlator in question have integral degree, via

$$\begin{aligned}
l_j &= q_j - (\Theta_j^{\rho^\alpha J} + \Theta_j^{\rho^\beta J} + \Theta_j^{\rho^\gamma J}) \\
&= q_j - \sum_{i=1}^N (\alpha_i + \beta_i + \gamma_i + 3) \varphi_j^{(i)} \\
&= -2q_j - a_k \varphi_j^{(k)} - \sum_{i=1}^N \gamma_i \varphi_j^{(i)} \\
&= -2q_j - a_k \varphi_j^{(k)} - \sum_{i=1}^N ((a_i - 1) - \delta_{i,k-1}(a_{k-1} - 1) - \delta_{i,k-2}) \varphi_j^{(i)} \\
&= -2q_j - a_k \varphi_j^{(k)} - (1 - 2q_j) + (a_{k-1} - 1) \varphi_j^{(k-1)} + \varphi_j^{(k-2)} \\
&= -1 + \delta_{j,k-1} - \delta_{j,k} \\
&= \begin{cases} 0 & \text{if } j = k-1 \\ -2 & \text{if } j = k \\ -1 & \text{else.} \end{cases}
\end{aligned}$$

By the index-zero axiom (Axiom 5), the non-vanishing three-point correlator is given by -1 times the X_{k-1} -degree of

$$\frac{\partial W}{\partial X_k} = X_{k-1}^{a_{k-1}} + a_k X_k^{a_k-1} X_{k+1},$$

so

$$\left\langle 1_{\rho_k^{(a_{k-1}-1)} J}, 1_{\rho_k J}, \mu \right\rangle = -a_{k-1}.$$

As in the preceding lemma, we have μ and ν necessarily in sectors with trivial fixed loci (i.e. not in the untwisted sector), so $\eta^{\mu\nu} = 1$.

Substituting into Equation (23), we conclude that

$$1_{\rho_k^{a_k-1} J} \star 1_{\rho_k J} = -a_{k-1} 1_{\rho_{k-2} \rho_{k-1}^{a_{k-1}-1} J},$$

as claimed. \square

For completeness, we address the case of two-variable loop potentials in Lemma 4.4 below. As already indicated, this result has been obtained independently by Acosta [A]. 4.1.

Lemma 4.4. *For $N = 2$, $k \in \{1, 2\}$,*

$$1_{\rho_k^{a_k} J} = -a_{k-1} 1_{\rho_{k-2} \rho_{k-1}^{a_{k-1}-1} J}.$$

Proof. The method of proof for the preceding lemma is not directly applicable here, because for a two-variable loop, $\rho_k^{a_{k-1}} J = \text{id}$, so the multiplicands used in the proof do not all lie in Neveu-Schwartz sectors and the index-zero axiom is not directly applicable.

However, if $a_k > 3$, so $\mathcal{H}_{\rho_k^{a_k-2} J}$ and $\mathcal{H}_{\rho_k^2 J}$ are Neveu-Schwartz sectors, the proof of Lemma 4.3 is easily amended to yield the same conclusion by considering the product $1_{\rho_k^{a_k} J} = 1_{\rho_k^{a_k-2} J} \star 1_{\rho_k^2 J}$.

So it remains only to consider the cases of two-variable loop potentials with one of the exponents (which we may take to be a_2) equal to 2 or 3.

For convenience of notation, we will use variables $x := x_1$ and $y := x_2$, and change the subscripts in the obvious way, so for example $\rho_x := \rho_1$ and $\rho_y = \rho_2$.

- $W = x^{a_x}y + xy^3$, with $a_x \geq 3$. To prove the lemma, we need to show

$$1_{\rho_y J}^{\star 3} = -31_{\rho_x^2 \rho_y J}.$$

(If $a_x = 3$, then by symmetry of the W , the corresponding relation will hold with x and y exchanged.) Using Corollary 2.1, we see that

$$1_{\rho_y J}^{\star 2} = \langle 1_{\rho_y J}, 1_{\rho_y J}, \mu_{\text{id}} \rangle \eta^{\mu, \nu} \nu_{\text{id}},$$

and

$$1_{\rho_y J}^{\star 3} = \langle 1_{\rho_y J}, 1_{\rho_y J}, \mu_{\text{id}} \rangle \eta^{\mu, \nu} \langle \nu_{\text{id}} 1_{\rho_y J}, 1_{\rho_y J} \rangle 1_{\rho_x^2 \rho_y J}.$$

The coefficient of $1_{\rho_x^2 \rho_y J}$ is, by the composition axiom (Axiom 6) equal to $\langle 1_{\rho_y J}, 1_{\rho_y J}, 1_{\rho_y J}, 1_{\rho_y J} \rangle$. The line bundle degrees for this correlator are

$$\begin{aligned} l_x &= 2q_x - 4\Theta_x^{\rho_y J} = 2\left(\frac{2}{8}\right) - 4\left(\frac{1}{8}\right) = 0, \\ l_y &= 2q_y - 4\Theta_y^{\rho_y J} = 2\left(\frac{2}{8}\right) - 4\left(\frac{5}{8}\right) = -2. \end{aligned}$$

So the correlator is given by -1 times the x -degree of $\partial W / \partial y$. *i.e.* $\langle 1_{\rho_y J}, 1_{\rho_y J}, 1_{\rho_y J}, 1_{\rho_y J} \rangle = -3$, as required.

- $W = x^2y + xy^3$. The vector space generators of the The composition axiom argument used to compute $1_{\rho_y J}^{\star 3}$ above applies here, yielding $1_{\rho_y J}^{\star 3} = -2(1_{\rho_x \rho_y J})$.

For degree reasons, we see that $\mathcal{H}_{W, G^{\max}}$ has a ring generator $\mu = \alpha x^2 dx \wedge dy + \beta y^2 dx \wedge dy \in \mathcal{H}_{\text{id}}$ that is not in the vector subspace generated by $1_{\rho_x J}^{\star 2} = \gamma x^2 dx \wedge dy + \delta y dx \wedge dy$. Here γ and δ are determined by the \star -product, and we seek α and β so that

$$\mu^2 = -3(1_{\rho_x J} \star \mu).$$

It turns out that the matrix of the pairing $\mathcal{H}_{\text{id}} \otimes \mathcal{H}_{\text{id}} \rightarrow \mathbb{C}$ is given by the symmetric matrix $-\frac{1}{6}A_W^{-1}$. Then, by the pairing axiom (Axiom 7), the desired relation is equivalent to

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -3 \begin{pmatrix} \gamma & \delta \end{pmatrix} A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Consider a non-zero vector v orthogonal to (γ, δ) with respect to the inner product with matrix A^{-1} on \mathbb{C}^2 . Putting $(\alpha, \beta) = (\gamma, \delta) + \lambda v$ and substituting into the above relation, we obtain the quadratic equation

$$\lambda^2 v^T A^{-1} v + 2 \begin{pmatrix} \gamma & \delta \end{pmatrix} A^{-1} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 0.$$

The coefficients in this equation are non-zero, as the vanishing of either of them would contradict non-degeneracy of the form A^{-1} . Either solution specifies μ , which is not a multiple of $1_{\rho_x J}^{(\star 2)}$ because the $\lambda \neq 0$.

With μ determined, the lemma follows.

- $W = x^2y + xy^2$. In this case, $G^{\max} = \langle J \rangle$, with $J = (e^{2\pi i/3}, e^{2\pi i/3})$. The sectors \mathcal{H}_J and $\mathcal{H}_{J^{-1}}$ are Neveu-Schwartz, respectively of minimal and maximal degree (\deg_+^A). The identity sector is $\mathcal{H}_{\text{id}} = \mathbb{C}[x dx \wedge dy, y dx \wedge dy]$, and the multiplication of the generators into $\mathcal{H}_{J^{-1}}$ is determined by the pairing axiom (Axiom 7), from which it is easy to see that

$$(x dx \wedge dy)^2 = -2(x dx \wedge dy)(y dx \wedge dy) = (y dx \wedge dy)^2.$$

These are precisely the defining relations for the generators of \mathcal{Q}_{W^T} , so the desired isomorphism holds.

□

Using associativity of A model multiplication to avoid the identity (Ramond) sector, it is easy to see that the mirror map is surjective. The dimension count of Lemma 1.7 then guarantees that the relations are generated by those in Lemma 4.3 if $N > 2$ or Lemma 4.4 if $N = 2$, from which the desired isomorphism follows.

4.1.3. *Chain Potentials:* $W = \sum_{i=1}^{N-1} X_i^{a_i} X_{i+1} + X_N^{a_N}$.

Since degree is additive under multiplication in \mathcal{Q}_{W^T} and in $\mathcal{H}_W^{G^{\max}}$, the isomorphism (19) of graded vector spaces suggests that the desired ring isomorphism

$$\mathcal{Q}_{W^T} \xrightarrow{\cong} \mathcal{H}_W^{G^{\max}} \quad (24)$$

should be induced by the map

$$\begin{aligned} \mathbb{C}[Y_1, \dots, Y_N] &\longrightarrow \mathcal{H}_W^{G^{\max}} \\ Y_i &\longmapsto 1_{\rho_i J}, \end{aligned} \quad (25)$$

which is extended to $\mathbb{C}[Y_1, \dots, Y_N]$ by multiplicativity. The following two lemmas show that $\mathcal{H}_{W, G^{\max}}$ is generated by the elements $1_{\rho_i J}$, subject to the relations

$$(1_{\rho_k J})^{\star a_k} + a_{k-1} 1_{\rho_{k-2} J} \star (1_{\rho_{k-1} J})^{\star(a_{k-1}-1)} = 0.$$

This means that the kernel of the mirror map is precisely the Jacobian ideal dW^T , yielding the desired isomorphism.

Remark. Note the assumption that $q_N < \frac{1}{2}$ is essential to our arguments, as we will use the fact that $\text{Fix}(\rho_N J)$ is trivial.

We proceed to prove the necessary lemmas.

Notation. Define

$$\rho^\alpha := \prod_{i=1}^N \rho_i^{\alpha_i}.$$

Lemma 4.5. *If $\alpha_i + \beta_i \leq a_i - 1$ for $i \in \{1, \dots, N\}$, and $\rho^\alpha J, \rho^\beta J$ and $\rho^{\alpha+\beta} J$ have trivial fixed loci, then*

$$1_{\rho^\alpha J} \star 1_{\rho^\beta J} = 1_{\rho^{\alpha+\beta} J}.$$

Proof. The argument here is practically identical to the one used to prove Lemma (4.2). \square

Lemma 4.6.

$$1_{\rho_N^{a_N-1} J} \star 1_{\rho_{N-1} J} = 0$$

Proof. Note this relation corresponds to the Jacobian relation $\frac{\partial W^T_{\text{chain}}}{\partial Y_N} = 0$.

By definition (Equation (6)),

$$1_{\rho_N^{a_N-1} J} \star 1_{\rho_{N-1} J} = \sum_{\mu, \nu} \left\langle 1_{\rho_N^{a_N-1} J}, 1_{\rho_{N-1} J}, \mu \right\rangle \eta^{\mu\nu} \nu.$$

For the three point correlator $\left\langle 1_{\rho_N^{a_N-1} J}, 1_{\rho_{N-1} J}, \mu \right\rangle$ to be non-zero, the line bundles $|\mathcal{L}_j|$ must have integral degree.

We know from Corollary 2.1 that there is at most one group element gJ for which $\mu \in \mathcal{H}_{gJ}$ yields a non-zero three point correlator. For the sector \mathcal{H}_{gJ} , let us consider the implication of integrality of the line bundles $|\mathcal{L}_j|$, for $j \in \{N, N-1\}$:

$$\begin{aligned} l_N &= q_N - \Theta_N^{\rho_N^{a_N-1} J} - \Theta_N^{\rho_{N-1} J} - \Theta_N^{gJ} \\ &= q_N - (a_N + 1)\varphi_N^{(N)} - \Theta_N^{gJ} \\ &= -1 - \Theta_N^{gJ}. \end{aligned}$$

For this to be integral, we require $\Theta_N^{gJ} \in \mathbb{Z}$, i.e. gJ fixes X_N . Furthermore,

$$\begin{aligned} l_{N-1} &= q_{N-1} - \Theta_{N-1}^{\rho_N^{a_N-1}J} - \Theta_{N-1}^{\rho_{N-1}J} - \Theta_{N-1}^{gJ} \\ &= q_{N-1} - (a_N + 1)\varphi_{N-1}^{(N)} - 3\varphi_{N-1}^{(N-1)} - \Theta_{N-1}^{gJ} \\ &= -\varphi_{N-1}^{(N-1)} - \Theta_{N-1}^{gJ}. \end{aligned}$$

For this to be integral, we require $\Theta_{N-1}^{gJ} = 1 - \varphi_{N-1}^{(N-1)} \notin \mathbb{Z}$, i.e. gJ does not fix X_{N-1} .

Since a chain potential fixes consecutive variables, we conclude that gJ has one-dimensional fixed locus, and consequently \mathcal{H}_{gJ} is empty, and the product vanishes as claimed. \square

Lemma 4.7. For $k \in \{2, \dots, N\}$,

$$1_{\rho_k J}^{\star(a_k)} = -a_{k-1} 1_{\rho_{k-2} \rho_{k-1}^{a_{k-1}-1} J}.$$

Proof. Note these relations correspond to the Jacobian relations $\frac{\partial W^T}{\partial Y_{k-1}} = 0$.

For $2 \leq k \leq N-1$, the proof proceeds exactly as in Lemma (4.3).

For $k = N$, we face the obstacle that $\rho_N^{a_N-1}J$ is a Ramond Sector, so we cannot use the index-zero axiom as before. We could realize $1_{\rho_N J}^{\star(a_N)}$ as the product of $1_{\rho_N J}^{\star 2}$ and $1_{\rho_N J}^{\star(a_N-2)}$, but this fails to avoid the Ramond sector when $a_N = 3$. Instead, we mimic the computation in [FJR1], where the composition axiom (Axiom 6) is used to determine the ring structure of $\mathcal{H}_{E_7, G^{\max}}$.

The reader may check using Corollary 2.1 that

$$1_{\rho_N^{(a_N-2)}J} \star 1_{\rho_N J} = \left\langle 1_{\rho_N^{(a_N-2)}J}, 1_{\rho_N J}, \mu 1_{\prod \rho_i^{\gamma_i} J} \right\rangle \eta^{\mu, \nu} \nu 1_{\rho_N^{a_N-1}J},$$

with

$$\gamma_i = \begin{cases} 0 & \text{if } i = N-1 \\ a_i - 2 & \text{if } i = N-2 \\ a_i - 1 & \text{else.} \end{cases}$$

Multiplying by $1_{\rho_N J}$, we see

$$1_{\rho_N J}^{\star a_N} = \left\langle 1_{\rho_N^{(a_N-2)}J}, 1_{\rho_N J}, \mu 1_{\prod \rho_i^{\gamma_i} J} \right\rangle \eta^{\mu, \nu} \left\langle \nu 1_{\rho_N^{a_N-1}J}, 1_{\rho_N J}, 1_{\prod \rho_i^{\gamma_i - \delta_{i,N}} J} \right\rangle 1_{\rho_{N-2} \rho_{N-1}^{a_{N-1}} J}.$$

Now, by the composition axiom,

$$\left\langle 1_{\rho_N^{(a_N-2)}J}, 1_{\rho_N J}, \mu 1_{\prod \rho_i^{\gamma_i} J} \right\rangle \eta^{\mu, \nu} \left\langle \nu 1_{\rho_N^{a_N-1}J}, 1_{\rho_N J}, 1_{\prod \rho_i^{\gamma_i - \delta_{i,N}} J} \right\rangle = \left\langle 1_{\rho_N^{(a_N-2)}J}, 1_{\rho_N J}, 1_{\rho_N J}, 1_{\prod \rho_i^{\gamma_i - \delta_{i,N}} J} \right\rangle.$$

Since all the sectors in this four-point correlator are Neveu-Schwartz, we may use the index-zero axiom to determine its value. A calculation similar to the other index-zero calculations yields for the degrees of the line bundles $|\mathcal{L}_j|$:

$$l_j = \begin{cases} -2 & \text{if } j = N \\ 0 & \text{if } j = N-1 \\ -1 & \text{else.} \end{cases}$$

So, the four-point correlator is -1 times the X_{N-1} degree of $\partial W / \partial X_N = a_N X_N^{a_N-1} + X_{N-1}^{a_{N-1}}$, namely $-a_{N-1}$. This completes the proof of Lemma 4.7. \square

Surjectivity of the mirror map is again clear from associativity of A-model multiplication, where we avoid Ramond sectors (so we can apply the preceding lemmas) by noting that $\rho^\gamma J$ has trivial fixed locus as long as $\gamma_N < a_{N-1}$. A dimension count using Lemma 1.7 then indicates that the relations in $\mathcal{H}_{W, G^{\max}}$ are generated by those in the lemmas, and the desired isomorphism follows.

4.2. SL symmetries for Calabi-Yau Loop Potentials. As evidence that the B -model multiplication defined in Section 2.2 is the appropriate product to consider in the context of LG-via-LG mirror symmetry, we prove the following theorem.

Theorem 4.8. *Let $W(X_1, \dots, X_N)$ be a loop potential with N odd, satisfying the Calabi-Yau condition: $\sum_i q_i = 1$. Let G be an admissible orbifold group such that $G \subset SL_N \mathbb{C}$. Then the mirror map (Equation (19)) is a Frobenius algebra isomorphism.*

Remark. Note that the group generated by the exponential grading operator J is automatically a subgroup of $SL_N \mathbb{C}$ in the Calabi-Yau case.

The theorem is applicable more generally than the statement initially suggests, as the FJRW A -model depends only on the charges and the orbifold group, not the presentation of the singularity [R1]. So, for example, the J -orbifolded A -models coincide for $W_{\text{loop}} = X_1^4 X_2 + X_2^4 X_3 + X_3^4 X_4 + X_4^4 X_5 + X_5^4 X_1$ and $W_{\text{Fermat}} = X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5$, and the J -orbifolded A -model of the latter may be computed as the SL -orbifolded B -model of the former.

Proof. Recall the because of the loop structure of the potential, the fixed locus for $g \in G$ is trivial unless $g = \text{id}$.

By Theorem 3.5, we know this map is a bijection. To see that it is an isomorphism of Frobenius algebras, we consider B -model multiplication between untwisted sectors, between twisted sectors, and between an untwisted sector and a twisted sector. *Untwisted B -model sector:*

$$\prod_{j=1}^N Y_j^{\alpha_j} dY_j \mid \text{id} \rangle \mapsto 1 \left| \prod_{j=1}^N \rho_j^{\alpha_j+1} \right\rangle,$$

where we note that since N is odd, the A -model sector corresponding to the monomial $\prod_{j=1}^N Y_j^{\alpha_j}$ is *not* the identity sector, so has trivial fixed locus.

Note that on the A -model side, the identity sector has degree $\hat{c} = N - 2 \sum q_i$, which is an odd integer, while the twisted sector corresponding to a group element $g \in SL_N \mathbb{C}$ has degree $2 \sum_i (\Theta_i^g - q_i)$, an even integer. Since degree is additive under multiplication, the product of two Neveu-Schwartz invariants has no component in the identity sector.

Consequently, in the A -model product (Equation (6)), all invariants appearing with non-zero coefficient on the right-hand side are Neveu-Schwartz invariants for the action of the maximal A -model symmetry group, and the correlators required to determine the multiplication are as computed in the subsection on Loop potentials in Section 4. *i.e.* The multiplicative relations on the A -model twisted sectors correspond precisely to the Jacobian relations in the B -model untwisted sector.

We must now consider the *Twisted B -model sectors*:

Since the B -model twisted sectors have trivial fixed loci, the mirror map sends them all to the A -model untwisted sector.

$$1 \left| \prod_{j=1}^N \bar{\rho}_j^{r_j+1} \right\rangle \mapsto \prod_{j=1}^N X_j^{r_j} dX_j \mid \text{id} \rangle,$$

On the A -model side, by the pairing axiom (Axiom 7),

$$\left(\prod_{j=1}^N X_j^{r_j} dX_j \right) 1_{\text{id}} \star \left(\prod_{j=1}^N X_j^{s_j} dX_j \right) 1_{\text{id}} = \left\langle \prod_{j=1}^N X_j^{r_j}, \prod_{j=1}^N X_j^{s_j} \right\rangle 1_{J^{-1}},$$

On the B -model side,

$$\left| \prod_{j=1}^N \bar{\rho}_j^{r_j+1} \right\rangle \star \left| \prod_{j=1}^N \bar{\rho}_j^{s_j+1} \right\rangle$$

vanishes unless every variable is fixed in $\left| \prod_{j=1}^N X_j^{r_j+s_j+1} dX_j \right\rangle$, which means precisely that $\prod_{j=1}^N X_j^{r_j+s_j} dX_j = \lambda \text{hess } W$ for some $\lambda \in \mathbb{C}$. *i.e.* the product is given by

$$\left| \prod_{j=1}^N \bar{\rho}_j^{r_j+1} \right\rangle \star \left| \prod_{j=1}^N \bar{\rho}_j^{s_j+1} \right\rangle = \frac{1}{\lambda} \left\langle \prod_{j=1}^N X_j^{r_j}, \prod_{j=1}^N X_j^{s_j} \right\rangle 1_{\text{id}},$$

and the products coincide up to a scalar factor for the A and B models.

It remains only to check that the multiplication between the twisted and untwisted B -model sectors satisfies the same relations as the corresponding A -model products. The B -model \mathcal{Q}_{id} -module structure means the only way such a product can be non-trivial is if the multiplicand from the untwisted sector is 1_{id} – the multiplicative identity. Since the mirror map preserves the identity, we need only show that on the A -model side,

$$\left| \prod_{j=1}^N \rho_j^{\alpha_j+1} \right\rangle \star \left| \prod_{j=1}^N X_j^{r_j} \right| \text{id} \rangle = 0.$$

This holds for degree reasons: the untwisted sector is the only sector with odd degree, and the twisted sectors all have even degree; by additivity of degree, the product has odd degree, so since it does not lie in the untwisted sector it must vanish. \square

Remark. The hypotheses for this theorem ensure that there are no non-zero contributions from the Ramond sector to products of Neveu-Schwartz invariants. The same argument will work in any case such a situation is established, so it should be possible to extend this result beyond the case of Calabi-Yau singularities orbifolded by subgroups of $SL_N \mathbb{C}$.

4.3. Strange Duality. Arnol'd's list of 14 exceptional singularities provides a source of interesting examples of Landau–Ginzburg Mirror Symmetry. In particular, we have the following:

Proposition 4.9. *Let W be one of the 14 exceptional unimodal singularities, and W^{SD} its Strange Dual. Then, there is a Frobenius algebra isomorphism*

$$\mathcal{H}_W^{\langle J \rangle} \cong \mathcal{Q}_{W^{\text{SD}}}.$$

Proof. Of course, when J generates G_W^{max} and $W^{\text{SD}} = W^T$, this is just a restatement of Theorem 4.1. However, examining Table 4.3, we see this is only the case for S_{12} , Z_{12} and E_{12} (which are self-dual), and Z_{11} and E_{13} (which are strange dual to each other).

To realize the observation for the remaining singularities in Arnol'd's list, we choose a different representative W' for each singularity W in such a way that

- $\mathcal{Q}_{W'} \cong \mathcal{Q}_W$.
- The charges of W' coincide with the charges of W , so $J_{W'} = J_W$.
- The maximal symmetry group of W' is generated by $J_{W'}$.
- Transposition yields the Strange Dual class in the updated list of exceptional singularities.

The Landau Ginzburg A -model \mathcal{H}_W^G constructed in [FJR1] is an invariant of $G \subset (\mathbb{C}^*)^N$ and the charges q_1, \dots, q_N [R1]. This means we are free to compute the FJRW ring of W orbifolded by $\langle J \rangle$ as $\mathcal{H}_{W'}^{\langle J \rangle}$.

Since W' is chosen so that J generates $G_{W'}^{\text{max}}$, we can then apply Theorem 4.1 to W' to yield the isomorphisms

$$\mathcal{H}_W^{\langle J \rangle} \cong \mathcal{H}_{W'}^{\langle J \rangle} \cong \mathcal{Q}_{(W')^T} \cong \mathcal{Q}_{W^{\text{SD}}}.$$

\square

One can attempt to use the original representative W where $\langle J \rangle$ is not necessarily G_W^{max} . Indeed, it fits into the general Landau–Ginzburg Orbifold Mirror Conjecture using the orbifold B -model of Section 2.2.

TABLE 1. Arnold's list of the 14 exceptional unimodal singularities W , with representatives W' chosen so Strange Duality is compatible with transposition.

| Class | W | $\langle J \rangle = G_W^{\max}$ | W' |
|----------|----------------------|----------------------------------|----------------------|
| Q_{10} | $x^2z + y^3 + z^4$ | Yes | $x^2z + y^3 + z^4$ |
| E_{14} | $x^2 + y^3 + z^8$ | No | $x^2 + y^3 + xz^4$ |
| Q_{11} | $x^2z + y^3 + yz^3$ | Yes | $x^2z + y^3 + yz^3$ |
| Z_{13} | $x^2 + y^3z + z^6$ | No | $x^2 + y^3z + z^3x$ |
| Q_{12} | $x^2z + y^3 + z^5$ | No | $x^2z + y^3 + xz^3$ |
| S_{11} | $x^2y + y^2z + z^4$ | Yes | $x^2y + y^2z + z^4$ |
| W_{13} | $x^2 + y^4 + yz^4$ | No | $x^2 + xy^2 + yz^4$ |
| S_{12} | $x^2y + y^3z + xz^2$ | Yes | $x^2y + y^3z + xz^2$ |
| U_{12} | $x^3 + y^3 + z^4$ | No | $x^2y + xy^2 + z^4$ |
| Z_{11} | $x^2 + y^3z + z^5$ | Yes | $x^2 + y^3 + yz^5$ |
| E_{13} | $x^2 + y^3 + yz^5$ | Yes | $x^2 + y^3z + z^5$ |
| Z_{12} | $x^2 + y^3z + yz^4$ | Yes | $x^2 + y^3z + yz^4$ |
| W_{12} | $x^2 + y^4 + z^5$ | No | $x^2 + xy^2 + z^5$ |
| E_{12} | $x^2 + y^3 + z^7$ | Yes | $x^2 + y^3 + z^7$ |

Example. We present here the case of U_{12} , which exhibits the general features of the other examples. We use Proposition 3.5 to show that we have a *Frobenius algebra isomorphism*

$$\mathcal{H}_{U_{12}}^{\langle J \rangle} \cong \mathcal{Q}_{U_{12}^T}^{\mathbb{Z}/3\mathbb{Z}},$$

rather than just the bi-graded vector space isomorphism guaranteed by theorem 3.5.

We know from the preceding discussion that

$$\mathcal{H}_{U_{12}}^{\langle J \rangle} \cong \mathcal{Q}_{U_{12}^{SD}} \cong \mathbb{C}[x, y, z] / \langle x^2, y^2, z^3 \rangle.$$

Remark. Note the isomorphism claimed between the Milnor rings of $U'_{12} = X^2Y + XY^2 + Z^4$ and $U_{12} = x^3 + y^3 + z^4$ is induced by the map

$$\mathbb{C}[X, Y, Z] \rightarrow \mathcal{Q}_{U_{12}}$$

which sends $X \mapsto \omega x + \omega^2 y$, $Y \mapsto \omega^2 x + \omega y$, and $Z \mapsto z$.

For the B -model of $U_{12}^T = x^3 + y^3 + z^4$, we note that the group dual to $\langle J \rangle$ is the SL subgroup of $G_{U_{12}^T}^{\max}$, namely $\mathbb{Z}/3\mathbb{Z}$ generated by $(\omega, \omega^2, 1)$.

$$\text{Fix}(\omega, \omega^2, 1)^k = \begin{cases} \mathbb{C}_{xyz}^3 & \text{if } k = 0 \\ \mathbb{C}_z & \text{if } k = 1, 2 \end{cases}, \quad \text{and} \quad \mathcal{Q}_{U_{12}^T}^{\mathbb{Z}/3\mathbb{Z}} = \begin{cases} \langle e_0, ze_0, z^2e_0, xye_0, xyz^2e_0 \rangle & \text{if } k = 0 \\ e_k & \text{if } k = 1, 2 \end{cases},$$

where $e_0 = dx \wedge dy \wedge dz$ and $e_1 = dz = e_2$.

We put $X = e_1$, $Y = e_2$ and $Z = ze_0$.

Note

$$\deg X = \deg Y = \frac{1}{2}(1 - 2q_x) + \frac{1}{2}(1 - 2q_y) = \frac{1}{3},$$

while

$$\deg Z = q_z = \frac{1}{4}.$$

We observe immediately that $Z^3 = 0$ (since multiplication in the untwisted sector is just multiplication in the unorbifolded Milnor ring).

Further, $X^2 = 0 = Y^2$, since the variables x and y are fixed in neither the $(\omega, \omega^2, 1)$ -sector, nor the $(\omega^2, \omega, 1)$ -sector.

Meanwhile, $XY = \alpha xy e_0$ for $\alpha \neq 0$, since xy has non-zero pairing with $\text{hess}(U_{12}|_{C_z}) = 12z^2$.

Thus we see the degree preserving map

$$\mathbb{C}[X, Y, Z] \mapsto \mathcal{Q}_{U_{12}^T}^{\mathbb{Z}/3\mathbb{Z}}$$

defined by $X \mapsto e_1$, $Y \mapsto e_2$ and $Z \mapsto ze_0$ is surjective, with kernel $\langle Z^3, X^2, Y^2 \rangle$, and induces an isomorphism

$$\mathcal{H}_{U_{12}}^{(J)} \cong \mathcal{Q}_{U_{12}} \mapsto \mathcal{Q}_{U_{12}^T}^{\mathbb{Z}/3\mathbb{Z}}.$$

REFERENCES

- A. P. Acosta, *FJRW rings and Landau–Ginzburg mirror symmetry in two dimensions.*, In preparation.
- AGV. V. Arnold, A. Gusein-Zade and A. Varchenko, *Singularities of differential maps*, vol I, II, Monographs in Mathematics.
- B. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, Integrable systems and algebraic geometry (Kobe / Kyoto, 1997), World Sci. Publishing, River Edge, NJ, 1998, pp. 1-32.
- BH. P. Berglund and T. Hübsch, *A Generalized Construction of Mirror Manifolds*, Nuclear Physics B, vol 393, 1993.
- BK. P. Berglund and S. Katz, *Mirror symmetry constructions: a review*, *Mirror Symmetry, II*, AMS/IP Stud. Adv. Math., vol. 1, Amer. Math. Soc., Providence, RI, 1997, pp. 87-113.
- CdK. P. Candelas, X. de la Ossa and S. Katz, *Mirror Symmetry for Calabi-Yau Hypersurfaces in Weighted \mathbb{P}^4 and Extensions of Landau Ginzburg Theory*, Nucl. Phys. B 450 (1995), no 1-2, 267-290.
- CR. A. Chiodo and Y. Ruan, *Private Communication*
- IV. K. Intriligator and C. Vafa, *Landau–Ginzburg orbifolds*, Nuclear Phys. B **339** (1990), no. 1, 95–120.
- FJR1. H. Fan, T. Jarvis and Y. Ruan, *The Witten equation, mirror symmetry and quantum singularity theory*, arXiv:math.0712.4021.
- FJR2. ———, *Geometry and analysis of spin equations*, Comm. Pure Applied Math 61(2008), 745-788.
- FJR3. ———, *The Witten equation and its virtual fundamental cycles*, arXiv:math/0712.4025.
- FS. H. Fan and Y. Shen, *Quantum ring of singularity $X^p + XY^q$* , arXiv:0902.2327v1.
- GP. B. Greene and R. Plesser *Duality in Calabi-Yau Moduli Space*, Nucl. Phys. B338 (1990) 15
- Ka1. R. Kaufmann, *Singularities with symmetries, orbifold Frobenius algebras and mirror symmetry*. **math.AG/0312417**.
- Ka2. ———, *Orbifold Frobenius algebras, cobordisms and monodromies*. Orbifolds in mathematics and physics (Madison, WI, 2001), 135–161, Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002.
- Ka3. ———, *Orbifolding Frobenius algebras*. Internat. J. Math. **14** (2003), no. 6, 573–617.
- KSc. A. Klemm and R. Schimmrigk, *Landau–Ginzburg string vacua*, Nuclear Physics B, Volume 411, Issues 2-3, 10 January 1994, Pages 559-583
- Ko. M. Kobayashi, *Duality of weights, mirror symmetry and Arnold’s strange duality*, Tokyo J. Math. 31 (2008), no. 1, 225–251.
- KP+. M. Krawitz, N. Priddis, *et. al.*, *FJRW Rings and Mirror Symmetry*, arXiv:math.0903.3220.
- K. M. Kreuzer, *The mirror map for invertible LG models*, Phys. Lett. B 328 (1994), no. 3-4, 312318
- KS. M. Kreuzer and H. Skarke, *On the classification of quasihomogeneous functions*, Comm. Math. Phys. 150 (1992), no. 1, 137–147.
- LVW. W. Lerche, C. Vafa, N. Warner, *Chiral rings in $N=2$ superconformal theories*, Nucl. Phys. B324 (1989) 427.
- Roan. S.S. Roan, *The geometry of Calabi-Yau orbifolds*, in: S.T Yau (Ed.), *Essays on mirror manifolds*, Intern. Press, Hong Kong, 1992, pp. 342-348.
- R1. Y. Ruan, *Private Communication*
- R2. ———, *Riemann Surfaces, Integrable Hierarchies and Singularity Theory*, In preparation.
- Wa1. C.T.C. Wall, *A note on symmetry of singularities*, Bull. London Math. Soc. **12** (1980), no. 3, 169-175.
- Wa2. C.T.C. Wall, *A second note on symmetry of singularities*, Bull. London Math. Soc. **12** (1980), no. 5, 347-354.
- Wi. E. Witten, *Phases of $N = 2$ Theories In Two Dimensions*, Nucl.Phys. B **403** (1993), 159-222.

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